Philip BOALCH, ENS, Soutenance HDR, 12/12/12

Rough Plan

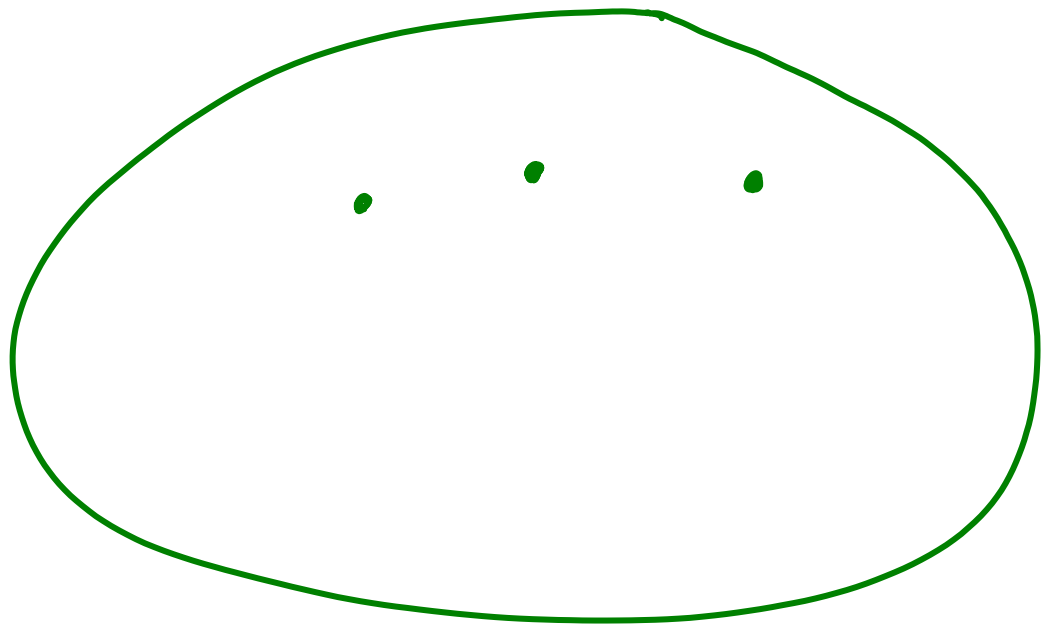
- Background
- Describe program
- Surprises
- Some more detailed statements

X a space, $x \in X$ a basepoint

$$\pi_1(X, x) = \left\{ \begin{array}{c} \text{homotopy classes of loops in } X \\ \text{based at } x \end{array} \right\}$$

– group under composition of loops

E.g. $X = \mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$ (m-punctured two sphere)



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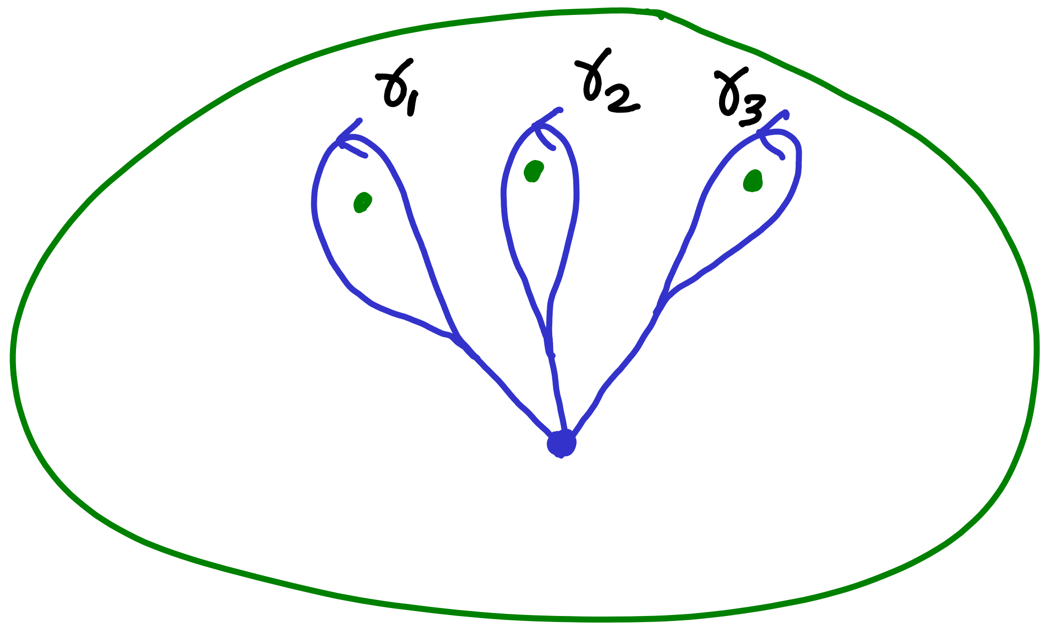

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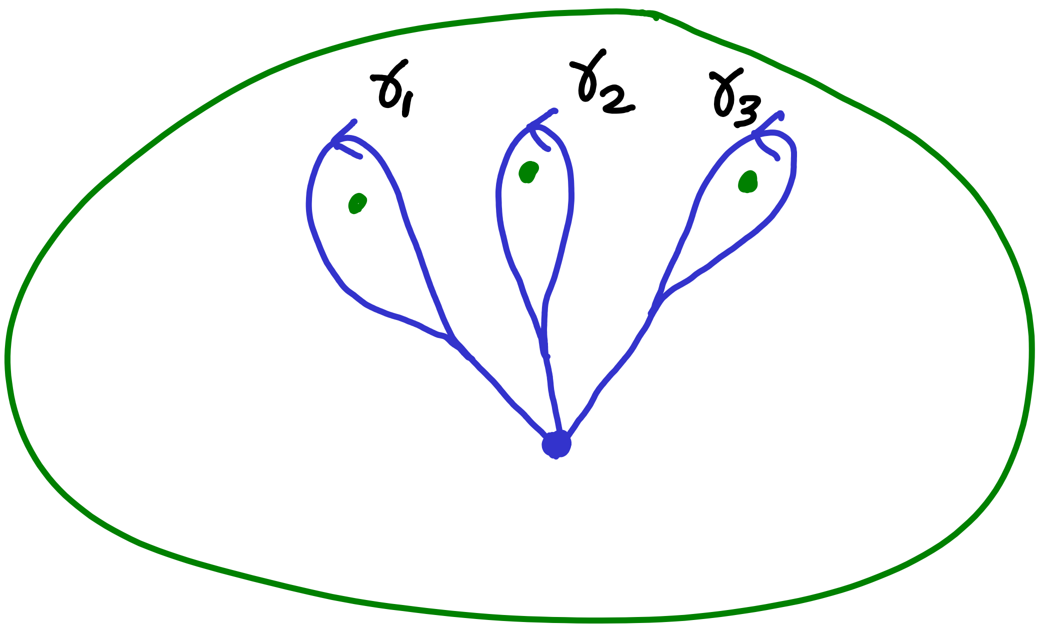
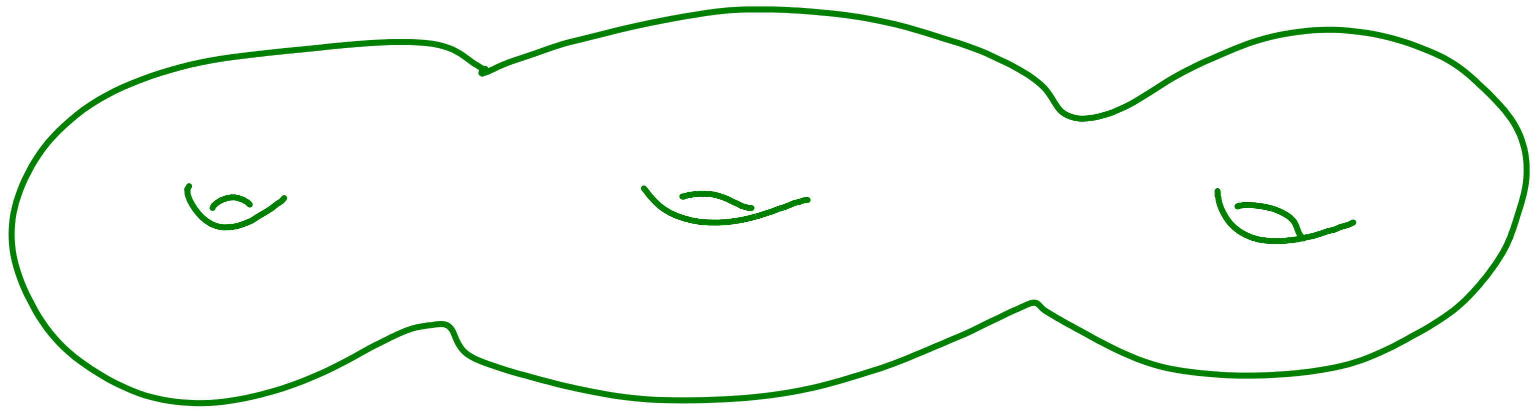


$$\pi_1(X, x) \cong \langle \gamma_1, \dots, \gamma_m \mid \gamma_1 \circ \cdots \circ \gamma_m = 1 \rangle$$

$$\cong \mathrm{Free}_{m-1} \quad \text{(Free group)}$$

m	0	1	2	3	4	5
π_1	1	1	$\mathbb{Z}$	Free_2	·······	

E.g. X = genus g compact Riemann surface



$$\pi_1(X) \cong \langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1 \rangle$$

where $[\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}$

E.g. $\pi_1 \cong \mathbb{Z}^2$ if $g = 1$

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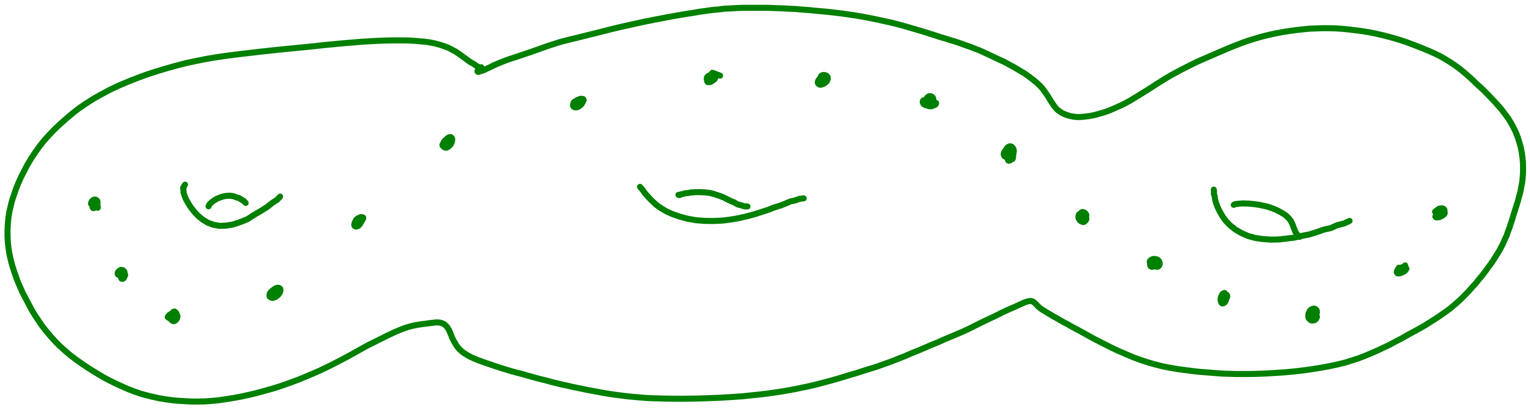

$$\pi_1(X) \cong \left\langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_m \;\middle|\; \prod_{i=1}^{g} [\alpha_i, \beta_i] \prod_{j=1}^{m} \gamma_j = 1 \right\rangle$$

"surface groups"

Nonabelian representations of surface groups arose in Riemann's work on the Gauss hypergeometric equation

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$$z(1-z)\,y'' + (az+b)\,y' + cy = 0$$

$[a, b, c$ constants, $\quad y(z)]$

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[a, b, c constants, $y(z)$]

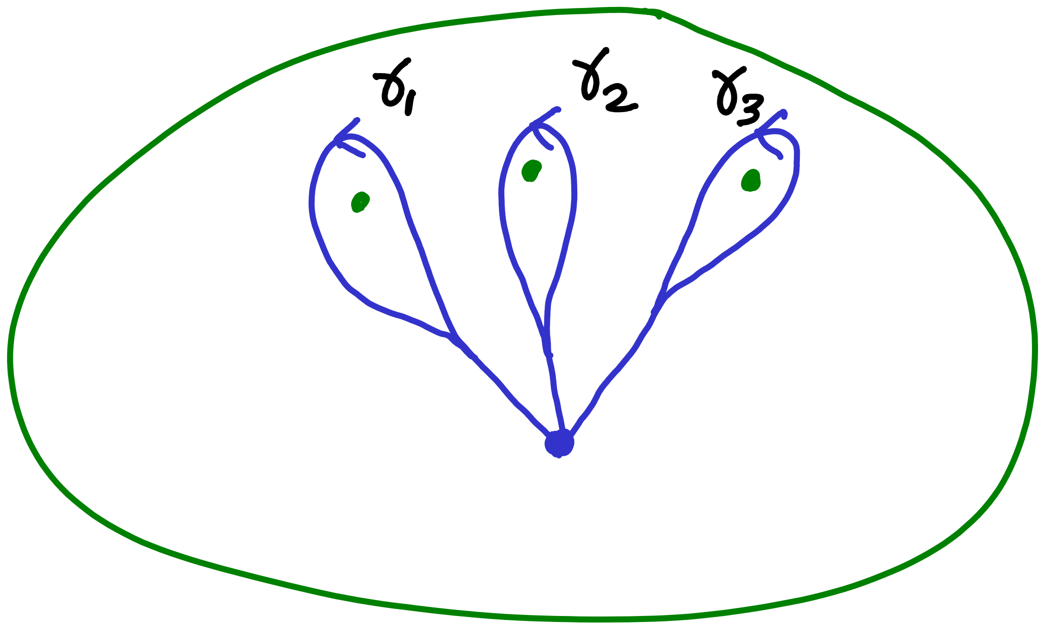
- second order linear algebraic differential equation
- singular points $0, 1, \infty \in \mathbb{P}^1(\mathbb{C})$

Riemann Have basis of solutions on any disk $U \subset X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

- Look at "monodromy" of bases of solutions around loops

$$\Rightarrow \quad \rho \in \mathrm{Hom}(\pi_1(X, x), GL_2(\mathbb{C}))$$

$$X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$$



$$M_i = \rho(\gamma_i)$$

$$\mathrm{Hom}(\pi_1(X), GL_2(\mathbb{C})) \cong \{ M_1, M_2, M_3 \in GL_2(\mathbb{C}) \mid M_1 M_2 M_3 = 1 \}$$

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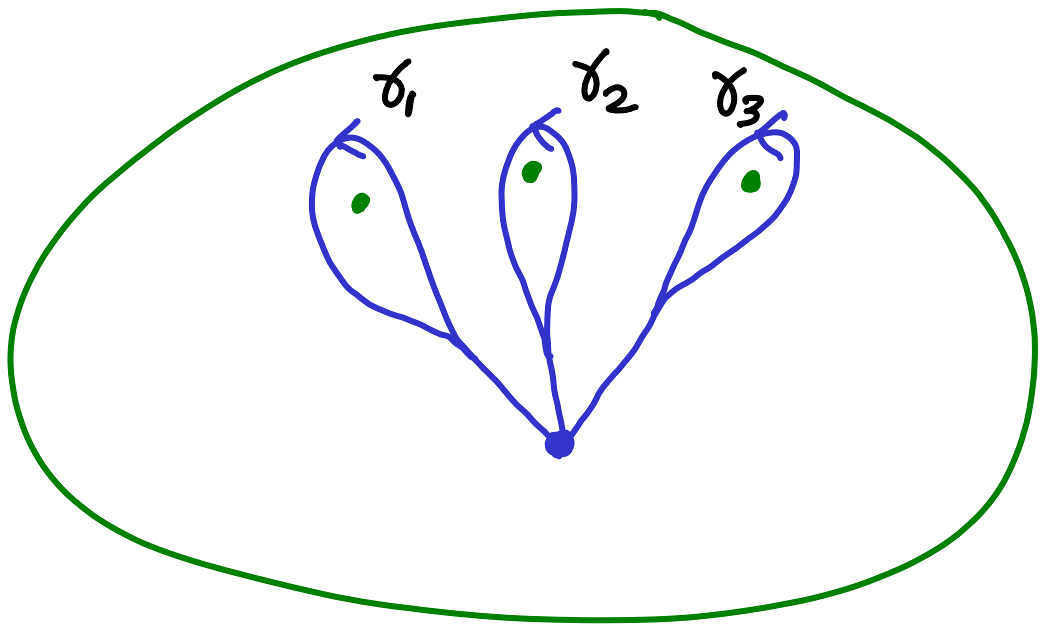

$$\mathrm{Hom}(\pi_1(X), GL_2(\mathbb{C})) \cong \{ M_1, M_2, M_3 \in GL_2(\mathbb{C}) \mid M_1 M_2 M_3 = 1 \}$$

- constants a, b, c ~ conjugacy classes of M_1, M_2, M_3
- conjugacy class of ρ in $\mathrm{Hom}(\pi_1, G)/G$ is intrinsic

(indept. of basepoint and initial basis)

More generally taking monodromy gives map:

Order n differential equations with singular points $a_1, \dots, a_m$

$$\Big\Downarrow \quad \text{"Riemann-Hilbert map"}$$

Point of $\quad \mathrm{Hom}\left(\pi_1\left(\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}\right), GL_n(\mathbb{C})\right) \big/ GL_n(\mathbb{C})$

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Hilbert's 21st problem (modern restatement):

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What's going on here?

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Hilbert's 21st problem (modern restatement):

What's going on here?

– is there a precise correspondence here somewhere?

Evolution 1

Order n differential equations

basis

cyclic vector

$n \times n$ first order systems $\frac{d}{dz} - A$

[A $n \times n$ matrix of mero. functions]

Evolution 2

$n \times n$ first order systems $\frac{d}{dz} - A$

connections on trivial rank n vector bundle $d - A\,dz$

(on $\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$)

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$$d - A\,dz = d - B$$

[B $n \times n$ matrix of mero. one-forms]

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$\|$

$d - B$

[B $n \times n$ matrix of mero. one-forms]

Locally have fundamental solutions $\Phi : U \to GL_n(\mathbb{C})$

$$\boxed{d\Phi = B\,\Phi}$$

Example $a_1, \dots, a_m \in \mathbb{C}$ $A_1, \dots, A_m \in \mathrm{End}(\mathbb{C}^n)$

$$d - \sum_1^m \frac{A_i}{z - a_i}\, dz$$

$\sum A_i = 0$ (no pole at ∞)

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$$\Big\Downarrow \ \mathrm{RH}$$

$$\rho \in \mathrm{Hom}\big(\pi_1(\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}), GL_n(\mathbb{C})\big)$$

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Theorem (Bolibruch)

This Riemann-Hilbert map is not surjective in general

Evolution 3

connections on trivial
rank n vector bundle
(on $\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$)

$\nabla = d - B$

connections ∇ on
rank n vector bundles V
(on $\Sigma \setminus \{a_1, \dots, a_m\}$)
Σ genus g Riemann surface

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$\nabla = d - B$

connections ∇ on rank n vector bundles V

(on $\Sigma \setminus \{a_1, \dots, a_m\}$)

Σ genus g Riemann surface

$\nabla : V \longrightarrow V \otimes \Omega^1$

$\nabla(fs) = (df)s + f(\nabla s)$

Locally: $\nabla = d - B$

$\Sigma = \overline{\Sigma} \setminus \{a_1, \ldots, a_m\}$ punctured Riemann surface

$G = GL_n(\mathbb{C})$

$\mathrm{Hom}(\pi_1(\Sigma), G) / G$

$\Sigma = \overline{\Sigma} \setminus \{a_1, \dots, a_m\}$ punctured Riemann surface

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$$\left\{ (V, \nabla) \;\middle|\; \begin{array}{l} V \to \Sigma \text{ holom. vector bundle} \\ \nabla \text{ holom. connection} \end{array} \right\} \Big/ \text{isom.} \xleftrightarrow[\text{analytic Riemann-Hilbert}]{\cong} \mathrm{Hom}(\pi_1(\Sigma), G) / G$$

$\Sigma = \overline{\Sigma} \setminus \{a_1, \dots, a_m\}$ punctured smooth algebraic curve$/\mathbb{C}$

$G = GL_n(\mathbb{C})$

Deligne–Plemelj

Riemann–Hilbert

$$\left\{ (V, \nabla) \;\middle|\; \begin{array}{l} V \to \Sigma \text{ alg. vector bundle} \\ \nabla \text{ alg. connection} \\ \text{with regular sing.s} \end{array} \right\} \Big/ \text{isom.} \xleftarrow{\;\;\simeq\;\;}\!\!\!\!\xrightarrow{\;\;\;\;} \mathrm{Hom}(\pi_1(\Sigma), G) / G$$

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$$\left\{ (\overline{V}, \overline{\nabla}) \;\middle|\; \begin{array}{l} \overline{V} \to \overline{\Sigma} \text{ alg. vector bundle} \\ \overline{\nabla} \text{ mero. connection} \\ \quad \text{with simple poles at } \{a_i\} \end{array} \right\} \Big/ \text{isom.}$$

$$\Big\downarrow \text{ restrict to } \Sigma$$

Deligne–Plemelj

Riemann–Hilbert

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– $\boxed{\text{Representations of } \pi_1 \text{ classify algebraic differential equations (in this sense)}}$

– similary for any smooth quasi-proj. var. (Deligne) $<$ add "flat/integrable"; simple poles $\rightsquigarrow$ logarithmic

– can now study transcendental aspects of RH map

E.g. $m = 0$ (no poles), Σ compact smooth complex alg. curve
$G = GL_n(\mathbb{C})$

$$\left\{ (V, \nabla) \;\middle|\; \begin{array}{l} V \to \Sigma \text{ alg. vector bundle} \\ \nabla \text{ alg. connection} \end{array} \right\} \Big/ \text{isom.} \quad \overset{RH}{\underset{\cong}{\longleftrightarrow}} \quad \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

$$\Big\downarrow \pi \text{ forget } \nabla$$

$$\left\{ \text{Alg. vector bundles } \underset{(\mathrm{rk}\, n,\, \deg 0)}{V \to \Sigma} \right\} \Big/ \text{isom.}$$

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$$\begin{tikzcd}
\left\{ (V,\nabla) \;\middle|\; \begin{array}{l} V \to \Sigma \text{ alg. vector bundle} \\ \nabla \text{ alg. connection} \end{array} \right\} \Big/ \text{isom.} \arrow[r, leftrightarrow, "RH", "\simeq"'] \arrow[d, "\pi"', "\text{forget } \nabla"] & \mathrm{Hom}(\pi_1(\Sigma), G) / G \\
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\end{tikzcd}$$

A. Weil: (1) π is not onto

(2) deformations of generic vector bundles ~ unitary representations

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\left\{ \text{Alg. vector bundles } \underset{(\mathrm{rk}\, n,\, \deg 0)}{V \to \Sigma} \right\} \Big/ \text{isom.} & & \mathrm{Hom}(\pi_1(\Sigma), U_n)/U_n \arrow[ll] \arrow[u, bend right]
\end{tikzcd}
$$

A. Weil: ① π is not onto

② deformations of generic vector bundles $\sim$ unitary representations

"Weil's unitary trick"

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\left\{ \text{Alg. vector bundles } V \to \Sigma \atop (\text{rk } n, \deg 0) \right\} \Big/ \text{isom.} & \mathrm{Hom}(\pi_1(\Sigma), U_n)/U_n \arrow[l] \arrow[u] \\
\left\{ \text{stable} \atop \text{alg. vector bundles } V \to \Sigma \right\} \Big/ \text{isom.} \arrow[u, phantom, "\cup"] \arrow[r, leftrightarrow, "\cong"] & \mathrm{Hom}^{irr}(\pi_1(\Sigma), U_n)/U_n \arrow[u, phantom, "\cup"]
\end{tikzcd}
$$

Mumford

Narasimhan – Seshadri

$V \to \Sigma$ is <u>stable</u> if

$$\frac{\deg(W)}{\operatorname{rank}(W)} < \frac{\deg(V)}{\operatorname{rank}(V)}$$

for any sub-bundle W of V

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(complex manifold + compatible symplectic structure)

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  (complex manifold + compatible symplectic structure)
- more generally $\operatorname{Hom}^{irr}(\pi_1(\Sigma), GL_n(\mathbb{C}))/GL_n(\mathbb{C})$ is hyperKähler
  (Hitchin, Donaldson, Corlette, Simpson)

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Here, only two complex structures are non-isomorphic:

1 as complex algebraic connections or complex π_1 representations

2 as a moduli space of stable Higgs bundles $\sim T^*\{\text{stable vector bundles}\}$

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Here, only two complex structures are non-isomorphic:

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$$(E, \Phi) \quad \begin{cases} E \to \Sigma & \text{holom. vector bundle} \\ \Phi : E \to E \otimes \Omega^1 & (\mathcal{O}\text{-linear}) \end{cases}$$

$$\Phi(fs) = f\,\Phi(s) \qquad \text{(degen. Leibniz)}$$

So in case $m=0$ have very rich picture

$$\begin{array}{ccccl}
 & & & \mathcal{M}_{Dol} & \text{Higgs bundles} \\
 & & & & \text{– Hitchin integrable systems} \\
 & & \nearrow & \updownarrow \ \text{Non-abelian Hodge} & \\
\Sigma & \Rrightarrow & \mathcal{M}(\Sigma) \ \rightarrow & \mathcal{M}_{DR} & \text{complex connections} \\
 & & \searrow & \updownarrow \ \text{RH} & \\
 & & & \mathcal{M}_{Betti} & \text{complex } \pi_1 \text{ rep.s}
\end{array}$$

complex curve

Hyperkahler manifold

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\text{complex curve} & \text{Hyperkahler manifold} & & \\
 & & \mathcal{M}_{Betti} & \text{complex } \pi_1 \text{ reps}
\end{tikzcd}$$

- Similarly for reps π_1 (punctured curve) (Simpson, Konno, Nakajima, ...)

Isomonodromy / Nonabelian Gauss-Manin connections

If $\begin{array}{ccc} \Sigma_p & \hookrightarrow & \Sigma \\ & & \downarrow \\ & & \mathbb{B} \end{array}$ family of smooth curves with marked points $/ \mathbb{B}$

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If
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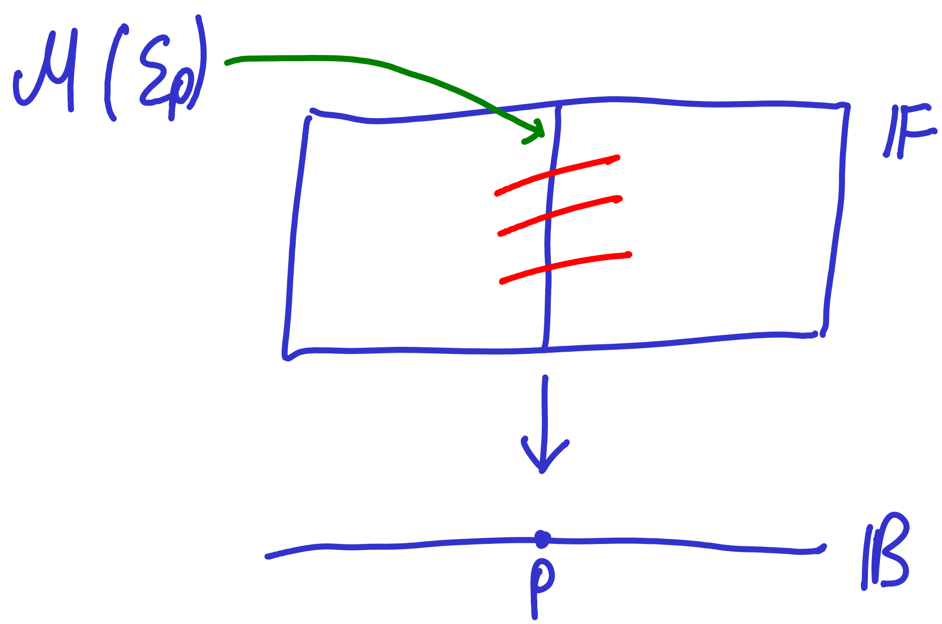
family of smooth curves with marked points / $\mathbb{B}$

then DeRham / Betti spaces fit into nonlinear fibration with natural complete flat (Ehresmann) connection



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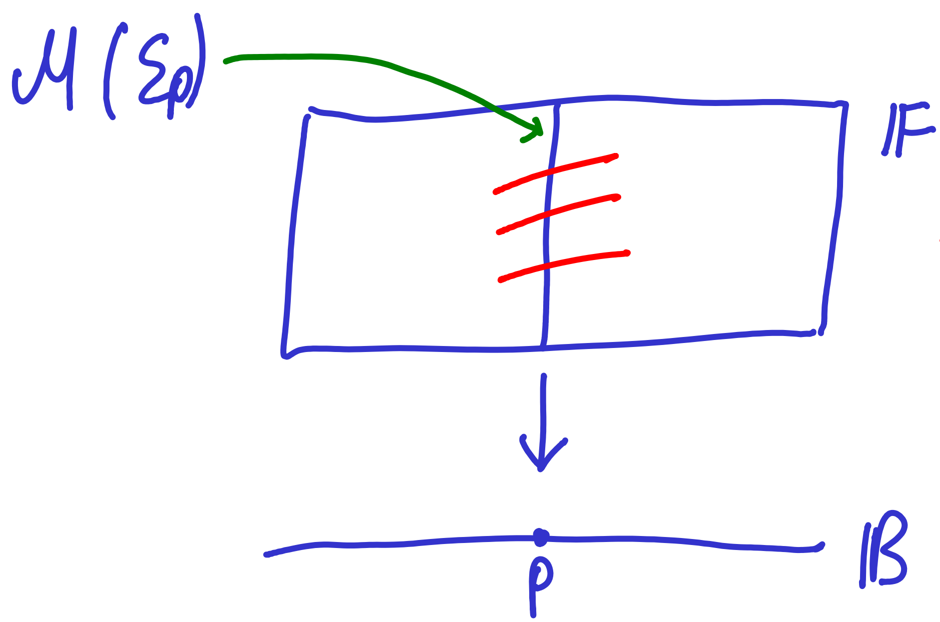

$\Rightarrow$ $\boxed{\begin{array}{l}\text{(nonlinear) monodromy:} \\ \pi_1(\mathbb{B}) \circlearrowright \mathcal{M}_B(\Sigma_p)\end{array}}$

Isomonodromy / Nonabelian Gauss-Manin connections

E.g. $\mathbb{B} = \mathbb{C}^m \setminus \text{diagonals} = \{ t_1, \dots, t_m \in \mathbb{C} \mid t_i \neq t_j \}$

$\Sigma_{\underset{\sim}{t}} = \mathbb{P}^1 \setminus \{ t_1, \dots, t_m \}$

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1 Nonlinear monodromy $\longleftrightarrow$ Hurwitz braiding

$$\beta_1 (M_1, \ldots, M_m) = (M_2, M_2^{-1} M_1 M_2, M_3, \ldots, M_m) \quad \text{etc.}$$

2 (DR) nonlinear connection $\longleftrightarrow$ Schlesinger's equations

$$d A_i = - \sum_{j \neq i} [A_i, A_j] \, d \log (t_i - t_j)$$

Isomonodromy / Nonabelian Gauss-Manin connections

Simplest case $m=4$, $G=GL_2(\mathbb{C})$ $\left(\dim_{\mathbb{C}} \mathcal{M}(\Sigma_{\underline{t}})=2\right)$

Schlesinger's equations $\Longleftrightarrow$ Painlevé $\overline{\underline{VI}}$ ODE (R. Fuchs)

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$$y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y'$$

$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{(t-1)}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

$t =$ cross ratio $(t_1, t_2, t_3\ t_4) \in \overline{\mathbb{B}} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

$\alpha, \beta, \gamma, \delta \in \mathbb{C}$ constants

"Nonlinear analogue of Gauss hypergeometric equation"

Isomonodromy / Nonabelian Gauss–Manin connections

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Schlesinger's equations $\Longleftrightarrow$ Painlevé $\underline{\mathrm{VI}}$ ODE (R. Fuchs)

$$\begin{array}{ccc} \mathcal{M}_{DR}(\Sigma_t) & \hookrightarrow & \mathbb{F} \\ & & \downarrow \end{array}$$

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"Nonlinear analogue of Gauss hypergeometric equation"

Algebraic solutions to Painlevé $\overline{\text{VI}}$ / Nonlinear Schwarz's List

Theorem (– 2002–2006)

There are 45 inequivalent sporadic algebraic solutions to P_{VI}

Algebraic solutions to Painlevé VI / Nonlinear Schwarz's List

Theorem (- 2002-2006)

There are 45 inequivalent sporadic algebraic solutions to P_{VI}

- 1 discrete family & 3 continuous families known earlier (Picard Okamoto Hitchin Dubrovin)
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		degree	genus	
1	Tetrahedral	6	0	Andreev–Kitaev
7	Octahedral	6–16	0,1	(2 by Kitaev)
33	Icosahedral	5–72	0,1,2,3,7	1 by Dubrovin; 2 by Dub.–Mazzocco; 2 by Kitaev
1	'Klein'	7	0	
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- Lisovyy–Tykhyy (arXiv 0809.4873, 2008) proved there are no others

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$$\boxed{\begin{aligned} &9(p^6q^2+p^2q^6) + 18p^4q^4 + 4(p^6+q^6) \\ &+26(p^4q^2+p^2q^4) + 8(p^4+q^4) + 57p^2q^2 \\ &+20(p^2+q^2) + 16 = 0 \end{aligned}}$$

$\Sigma = \overline{\Sigma} \setminus \{a_1, \dots, a_m\}$ punctured smooth algebraic curve$/\mathbb{C}$

$G = GL_n(\mathbb{C})$

$$\left\{ (\overline{V}, \overline{\nabla}) \;\middle|\; \begin{array}{l} \overline{V} \to \overline{\Sigma} \text{ alg. vector bundle} \\ \overline{\nabla} \text{ mero. connection} \\ \quad \text{with simple poles at } \{a_i\} \end{array} \right\} \Big/ \text{isom.}$$

restrict to Σ $\downarrow$

$$\left\{ (V, \nabla) \;\middle|\; \begin{array}{l} V \to \Sigma \text{ alg. vector bundle} \\ \nabla \text{ alg. connection} \\ \quad \text{with regular sing.s} \end{array} \right\} \Big/ \text{isom.} \quad \xleftrightarrow[\text{Riemann-Hilbert}]{\text{Deligne-Plemelj}}{\cong} \quad \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

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Irregular connections

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$y = e^z$ $\qquad\qquad$ $dy = y\,dz$

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- equations of Bessel, Airy, Whittaker, Kummer, ...
- Baker functions are solutions of irregular connections

Irregular connections

Local formal classification: (Hukuhara–Turrittin)

After passing to a finite cover any meromorphic connection on a vector bundle on a curve is formally meromorphically isomorphic to

$$D = d - \left(dQ + \Lambda \frac{dz}{z}\right)$$

where
- $Q = \dfrac{A_r}{z^r} + \cdots + \dfrac{A_1}{z}$, A_i diagonal $n \times n$ matrices
- $\Lambda \in \mathrm{End}(\mathbb{C}^n)$ commutes with $A_1, \dots, A_r$

[via $\mathbb{C}[\![z]\!][z^{-1}]$ valued gauge transformations]

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"Irregular type" "formal residue"

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Can describe ? in terms of fundamental groupoid of a related curve ...

Program

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- Study what happens when curve, marked points, ... vary and the resulting isomonodromic deformation equations & braiding
- Try to classify resulting "wild nonabelian Hodge" moduli spaces

Surprises

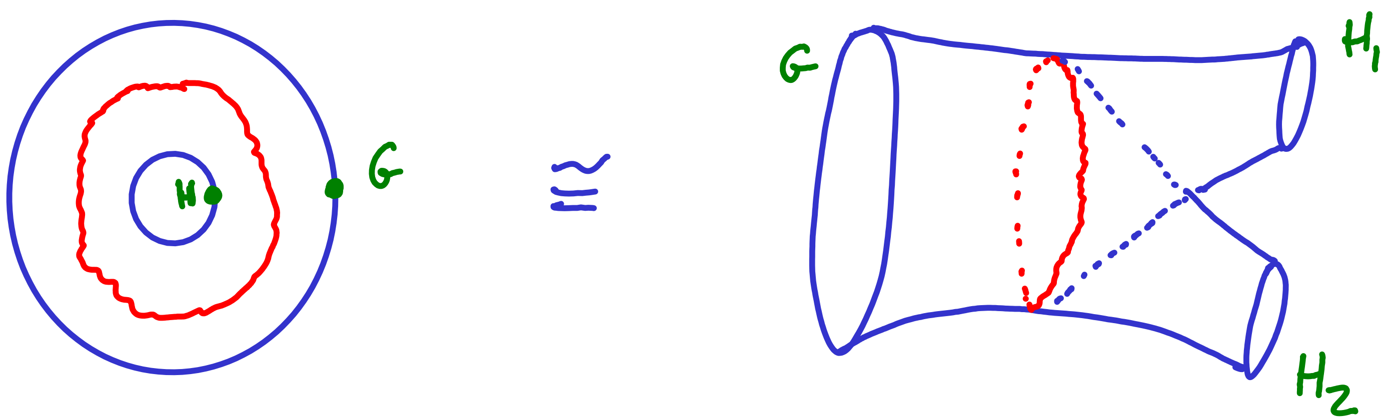
- Drinfeld/Semenov-Tian-Shansky's dual Poisson Lie group G^* appears as simple space of Stokes data
  - $\longrightarrow$ $U_q\mathfrak{g}$ quantises a space of Stokes data
  - $\longrightarrow$ geometric origins of quantum Weyl group
  - $\longrightarrow$ dual exponential maps $\boxed{\mathfrak{g}^* \longrightarrow G^*}$

$$
\begin{tikzcd}
 & U_q\mathfrak{g} \arrow[dl, "\text{DeConcini-Kac-Procesi}"'] \arrow[dr] & \\
\mathrm{Fun}(G^*) \arrow[dr] & & U\mathfrak{g} \arrow[dl] \\
 & \mathrm{Fun}(\mathfrak{g}^*) = \mathrm{Sym}(\mathfrak{g}) &
\end{tikzcd}
$$

Drinfeld-Jimbo

Surprises

- There is a "fission" operation (distinct from fusion) enabling inductive construction of wild character varieties



$H \cong H_1 \times H_2$

- fusion $\rightsquigarrow$ induction with respect to genus and no. of poles
- fission $\rightsquigarrow$ induction with respect to order of poles

Surprises

- Many known noncompact hyperkähler manifolds are canonically open submanifolds of some wild nonabelian Hodge moduli space $\mathcal{M}(\Sigma)$

(metrics are different $\rightsquigarrow$ view $\mathcal{M}(\Sigma)$ as "more transcendental version" of known spaces $\mathcal{M}^* \subset \mathcal{M}$)

E.g. ALE spaces $A_0, A_1, A_2, A_3, D_4, E_6, E_7, E_8$

ALF space D_2

Coadjoint orbits $\mathcal{O} \subset \mathfrak{gl}_n(\mathbb{C})$

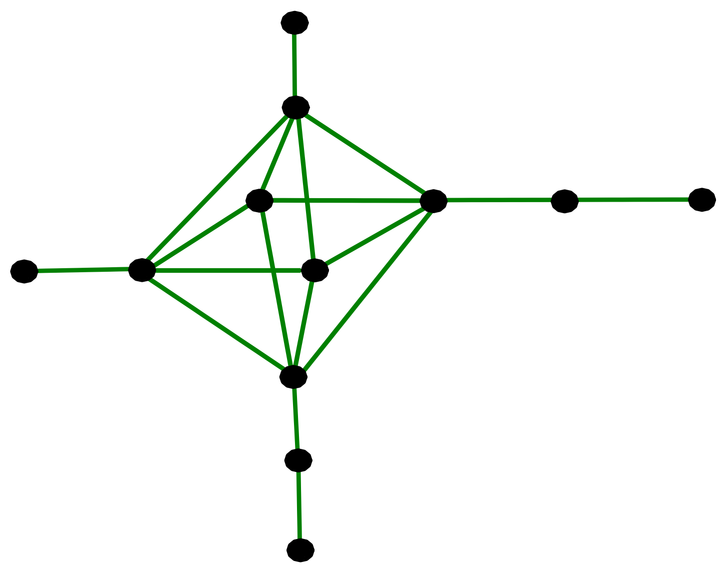
cotangent reductions $T \backslash\!\backslash T^* G /\!/ T \qquad G = GL_n(\mathbb{C})$

(Nakajima quiver varieties for supernova graphs)

Surprises

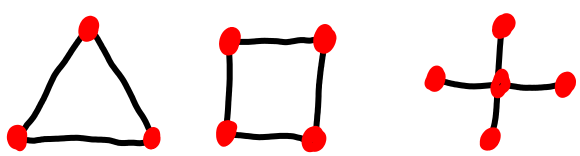
- There is a nice theory of Dynkin diagrams for many of these moduli spaces

"Supernova graphs"



generalising:

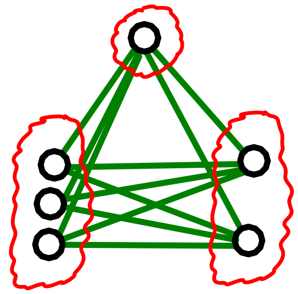
① Okamoto's affine Weyl group symmetries of Painlevé equations



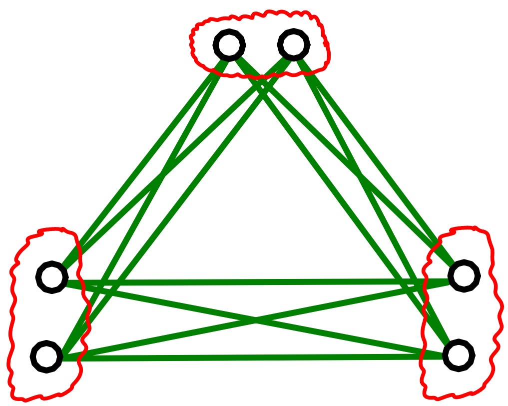
$\Longleftrightarrow$ P_{IV}, P_{V}, P_{VI}

② the 'star-shaped' case

Complete k partite graphs $\Longleftrightarrow$ Integer partitions with k parts

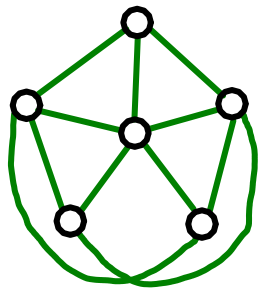


$\Longleftrightarrow$ $1+2+3=6$



$\Longleftrightarrow$ $2+2+2=6$

Complete k partite graphs $\Longleftrightarrow$ Integer partitions with k parts



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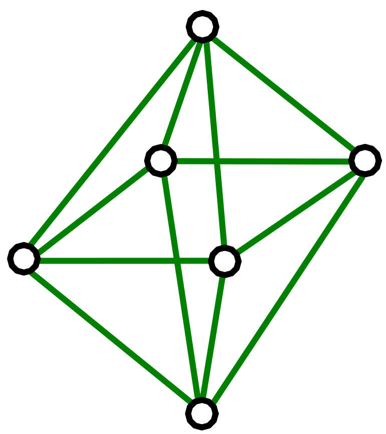

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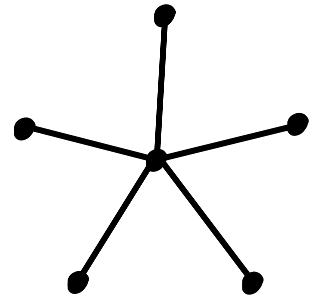
Graphs from partitions of $N \leq 6$

(omitting totally disconnected graphs $\mathcal{G}(n)$, and stars $\mathcal{G}(n,1)$)



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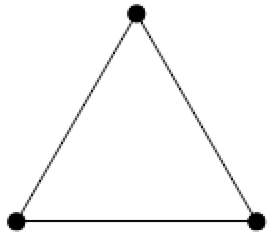

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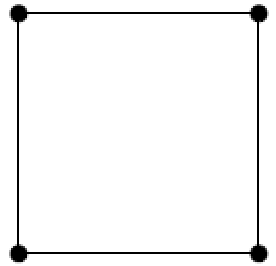

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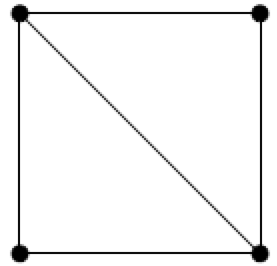

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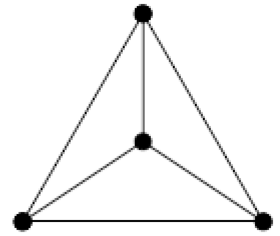

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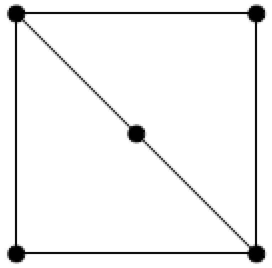

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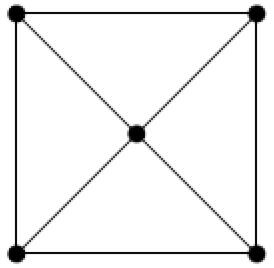

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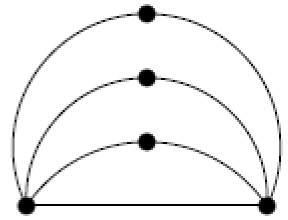

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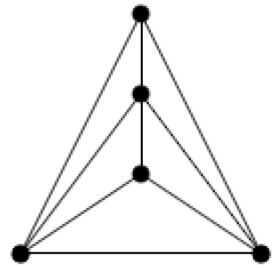

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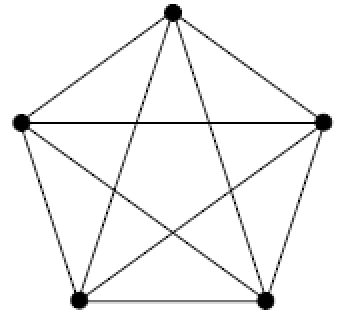

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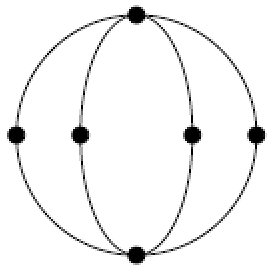

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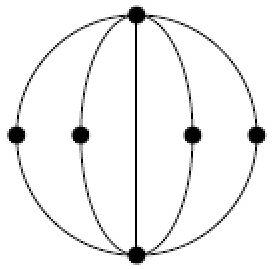

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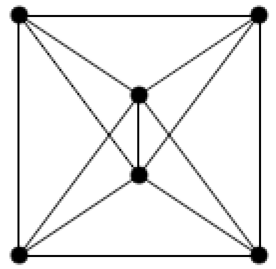

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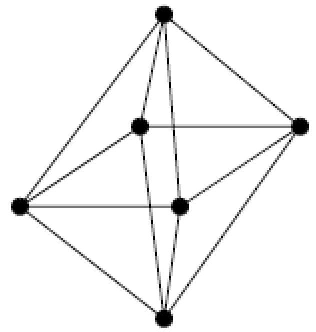

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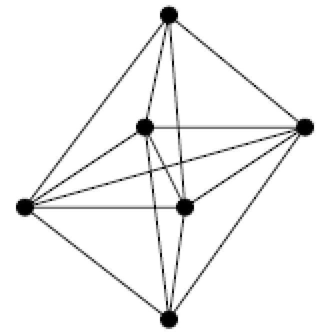

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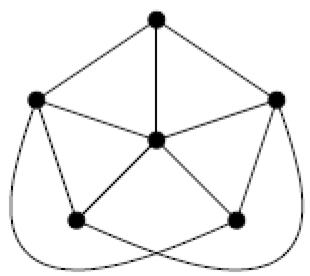

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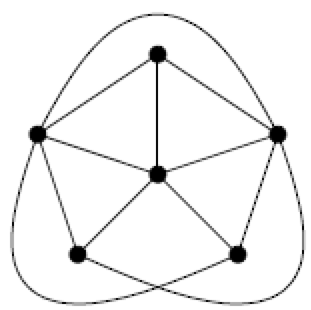

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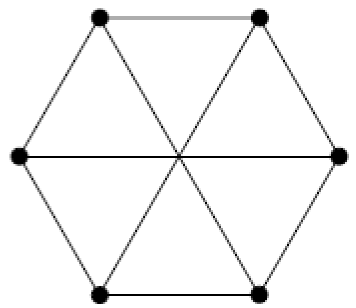

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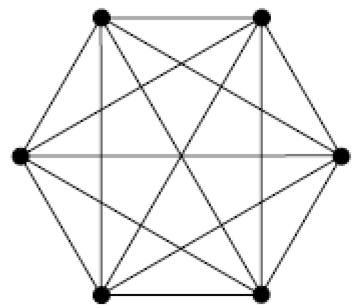

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Wild nonabelian Hodge theory on curves

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- adjoint orbits $\mathcal{O}_i \subset \mathfrak{l}_i$

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$L_i = C_{\mathfrak{g}}(Q_i, \theta_i) \subset H_i \subset \mathfrak{g}$

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Note that $\theta \in \mathfrak{t}_{\mathbb{R}}$ determines a parabolic $P_\theta \subset \mathfrak{g}$

$$P_\theta(\mathfrak{g}) = \mathrm{Stab}(\mathcal{F}_\theta), \quad (\mathcal{F}_\theta)_\alpha = \bigoplus_{\beta \geq \alpha} E_\beta \quad \left(\begin{array}{c}\text{eigenspaces}\\ \text{of } \theta\end{array}\right)$$

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& similarly $\mathfrak{p}_{\theta_i}(\mathfrak{h}_i) \subset \mathfrak{h}_i$ & $\mathfrak{l}_i$ is Levi of $\mathfrak{p}_{\theta_i}(\mathfrak{h}_i)$

Consider triples $(V, \nabla, \mathcal{F})$

- $V \longrightarrow \Sigma$ rank n holom. vector bundle
- $\nabla : V \longrightarrow V \otimes \Omega^1(*D)$ mero. connection $D = \sum a_i$
- $\mathcal{F} = (\mathcal{F}_i)_{i=1}^m$ flags in fibres $V_{a_1}, \ldots, V_{a_m}$

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- $\mathcal{F}_i \cong$ standard flag $\mathcal{F}_{\theta_i}$

Consider triples $(V, \nabla, \mathcal{F})$

- $V \to \Sigma$ rank n holom. vector bundle
- $\nabla : V \to V \otimes \Omega^1(*D)$ mero. connection $D = \sum a_i$
- $\mathcal{F} = (\mathcal{F}_i)_{i=1}^m$ flags in fibres $V_{a_1}, \dots, V_{a_m}$

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- $\pi(\Lambda_i) \in \mathcal{O}_i \subset \mathfrak{l}_i$ $\quad$ ($\pi : \mathfrak{p}_{\theta_i}(\mathfrak{h}_i) \twoheadrightarrow \mathfrak{l}_i$)

Thm (Biquard–B. '04 building on Hitchin, Donaldson, Corlette, Simpson, Simpson, Nakajima, Sabbah, ...)

The moduli space $\mathcal{M}_{DR}(\Sigma, \underset{\sim}{\theta}, \underset{\sim}{\alpha})$ of isomorphism classes of such mero. connections which are stable and parabolic degree zero is

- a hyperkähler manifold
- canonically diffeo. to a space of mero. Higgs bundles
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- Higgs fields should look like $-\frac{1}{2} dQ_i + \Lambda_i \frac{dz}{z} + \text{holom.}$ near a_i
- same 'rotation' of the weights/eigenvalues as in Simpson 1990

$$\begin{tikzcd} & & \mathcal{M}_{\mathrm{Dol}} \\ \Sigma \arrow[r, Rrightarrow] & \mathcal{M}(\Sigma) \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{M}_{\mathrm{DR}} \\ & & \mathcal{M}_{\mathrm{Betti}} \end{tikzcd}$$

Σ: irregular curve

$\mathcal{M}(\Sigma)$: Hyperkahler manifold

$\mathcal{M}_{\mathrm{Dol}}$: mero. Higgs bundles

meromorphic Hitchin integrable systems

(Bottacin–Markman)

$\mathcal{M}_{\mathrm{DR}}$: mero. connections

$\mathcal{M}_{\mathrm{Betti}}$: wild character varieties

Wild character varieties

(G = connected complex reductive gp)

$$\Sigma \longmapsto \mathrm{Hom}_{\mathbb{S}}(\Pi, G)/\underset{\sim}{H}$$

Irregular curve

Poisson variety

Irregular Betti spaces

Irreg RH on curves worked out decades ago for $G = GL_n(\mathbb{C})$

(Birkhoff Balser Jurkat Lutz Malgrange Sibuya Deligne Martinet Ramis ...)

- will give explicit as possible approach using groupoids (for any reductive G)

Irregular Betti spaces

Let Σ be an irreg. curve (marked points $a_1, \dots, a_m$, irreg. types $Q_1, \dots, Q_m$)

Let $\hat{\Sigma} \longrightarrow \Sigma$ be real oriented blow up of Σ at a_i

(each a_i replaced by a circle ∂_i, so $\partial \hat{\Sigma} = \partial_1 \sqcup \dots \sqcup \partial_m$)

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Then each Q_i determines:

1) A connected complex reductive group $H_i \subset G$

2) A finite set $\mathbb{A}_i \subset \partial_i$ of singular directions at a_i

and for each $d \in \mathbb{A}_i$

3) A unipotent group $\mathrm{Sto}_d(Q_i) \subset G$ normalised by H_i

1) H_i = stabilizer of Q_i under adjoint action

$\left(H_i = \{ g \in G \mid \mathrm{Ad}_g(A_i) = A_i \ \ \forall i \} \right)$

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so $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid [Y, X] = \alpha(Y) X \ \forall Y \in \mathfrak{t} \}$

Let $q_\alpha = \alpha \circ Q$ (mero. function near $a \in \Sigma$)

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then $d \in \partial$ is a singular direction supported by $\alpha \in R$

if $\exp(q_\alpha)$ has maximal decay as $z \to a$ along d

(leading term of q_α is real and negative along d)

& $\mathbb{A} \subset \partial$ is set of all sing. directions $(\forall \alpha \in R)$

3) Let $R(d) = \{ \alpha \mid \alpha \text{ supports } d \} \subset R$

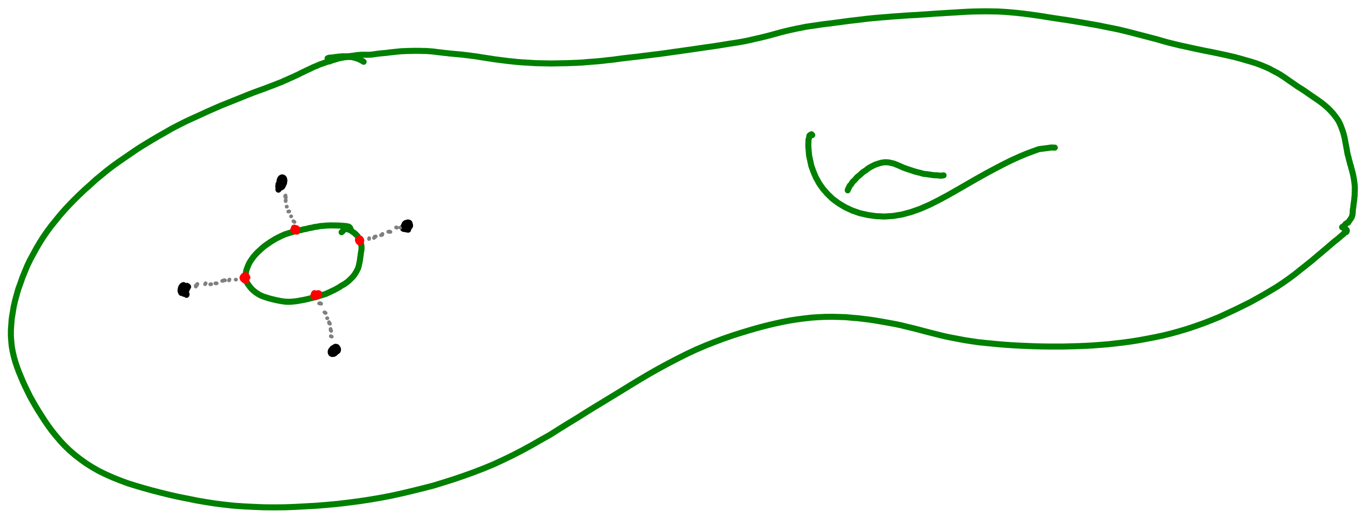
$$\mathrm{Sto}_d = \prod_{\alpha \in R(d)} \exp(\mathfrak{g}_\alpha) \hookrightarrow G$$

Lemma Sto_d is a well defined unipotent subgroup of G

Now puncture $\hat{\Sigma}$ once in its interior near each singular direction $d \in \mathbb{A}_i$, $i=1,\dots,m$

and let $\tilde{\Sigma} \subset \hat{\Sigma}$ be resulting punctured surface

$\tilde{\Sigma}$



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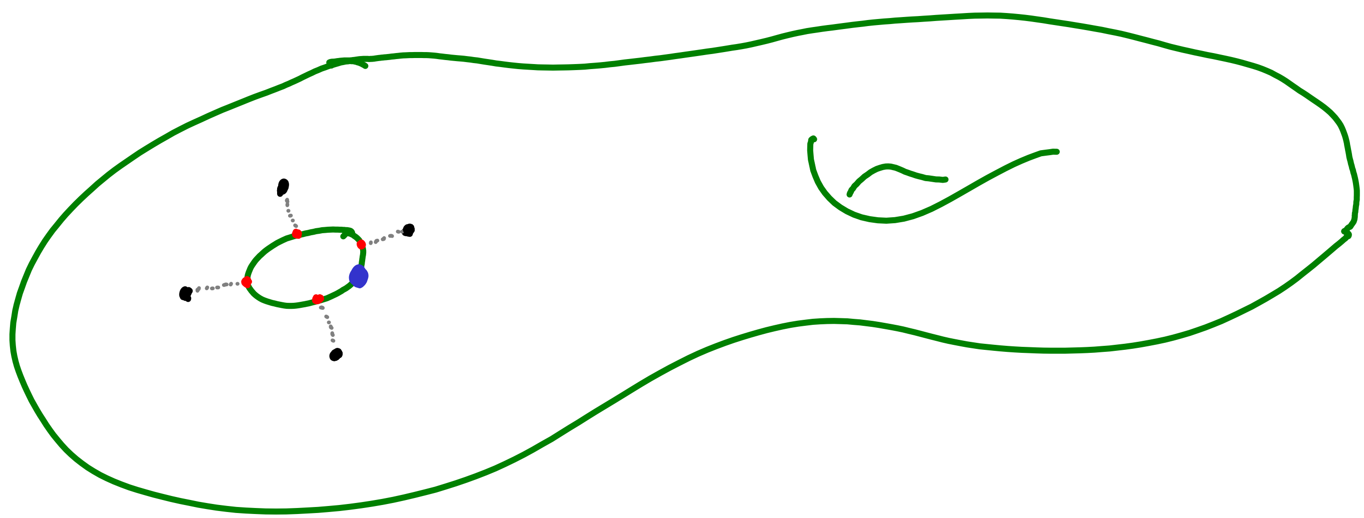

Choose a base point $b_i \in \partial_i$ in each boundary circle

Let $\Pi = \Pi_1(\tilde{\Sigma}, \{b_1, \dots, b_m\})$

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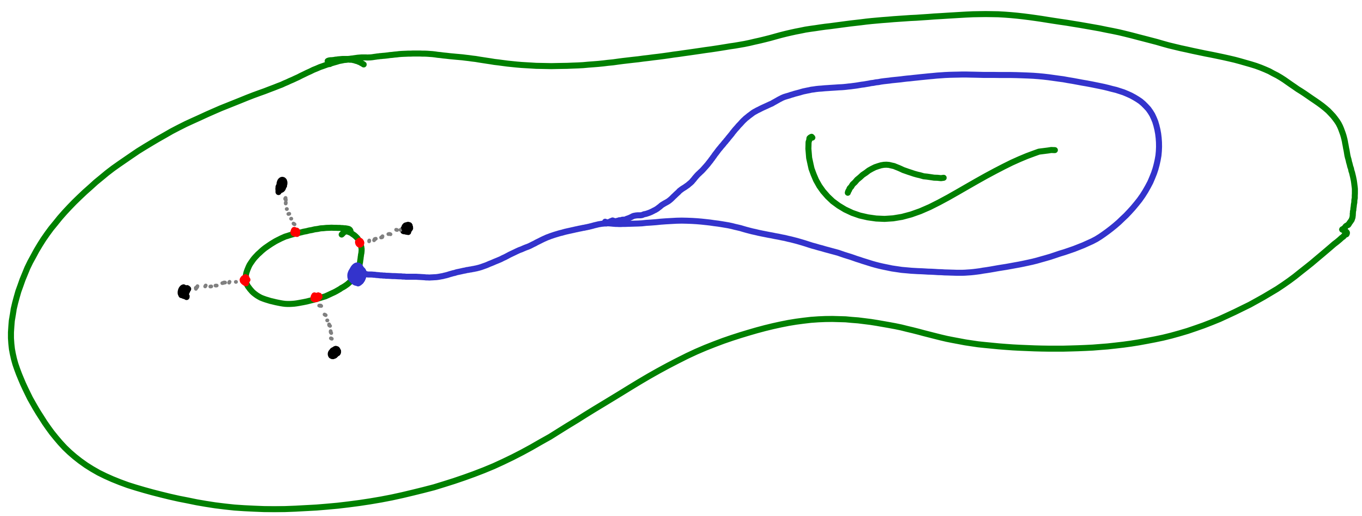

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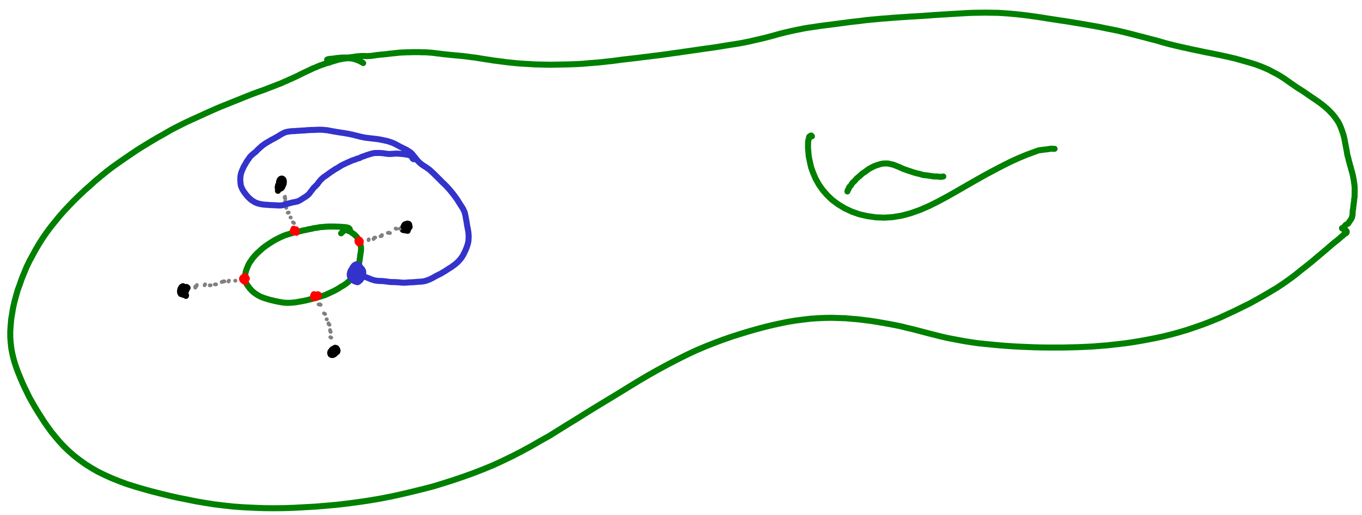

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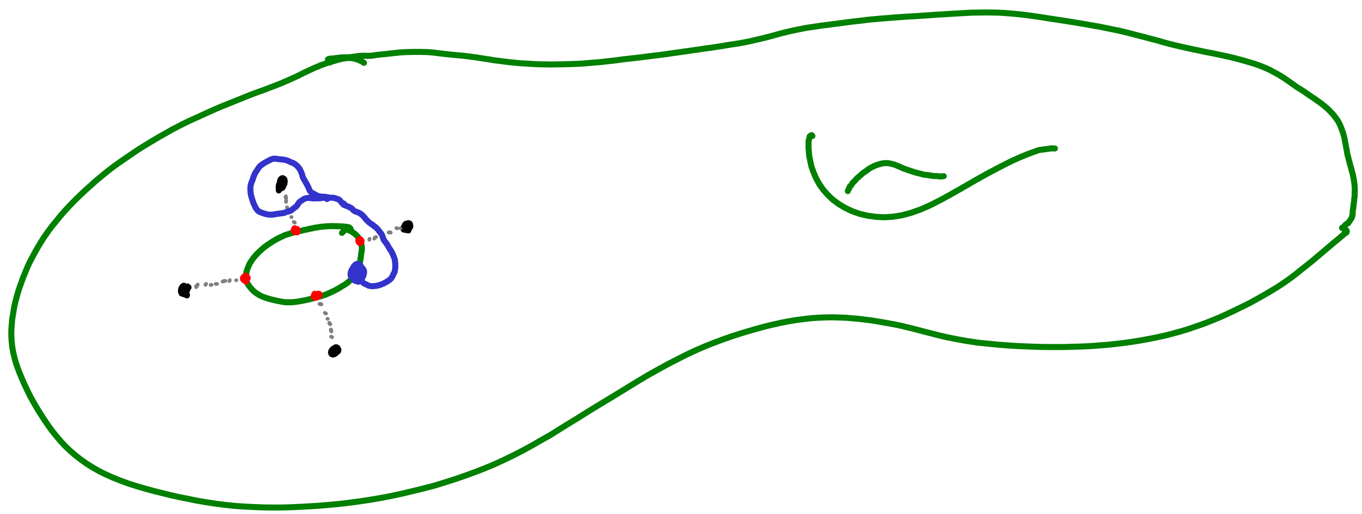

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$\cup$

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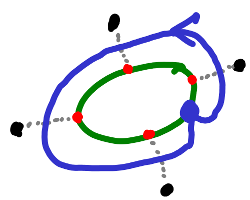

Now consider
$$\begin{array}{c}\mathrm{Hom}(\pi, G)\\ \cup\\ \mathrm{Hom}_{\mathbb{S}}(\pi, G)\end{array}$$
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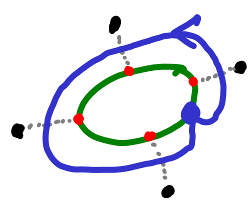
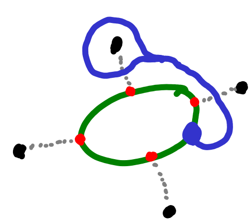


2) If γ goes around ∂_i from b_i until $d \in \mathbb{A}_i$ then loops around the corresponding puncture before returning to b_i, then $\rho(\gamma) \in \mathbb{S}\mathrm{to}_d$



Theorem (building on Birkhoff Balser Jurkat Lutz Malgrange Sibuya Deligne Martinet Ramis ...)

$$\left\{ (V,\nabla) \,\middle|\, \begin{array}{l} V \to \Sigma^\circ \text{ alg. } G\text{-bundle} \\ \nabla \text{ alg. connection s.t.} \\ \text{irreg. type } Q_i \text{ at } a_i \end{array} \right\} \Big/ \text{isom.} \quad \xleftrightarrow[\cong]{\text{Riemann-Hilbert-Birkhoff}} \quad \mathrm{Hom}_{\mathbb{S}}(\Pi, G) / \underset{\sim}{H}$$

where $\underset{\sim}{H} = H_1 \times \cdots \times H_m$

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- If all $Q_i = 0$ $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)/\underset{\sim}{H} = \mathrm{Hom}(\Pi, G)/G^m$

$$\cong \mathrm{Hom}(\pi_1(\Sigma \setminus \{a_i\}), G)/G$$

Better statement (building on Birkhoff Balser Jurkat Lutz / Malgrange Sibuya Deligne Martinet Ramis ...)

$(V,\nabla) \,\Big|\,$ $V \to \Sigma^\circ$ alg. G-bundle
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Stokes G-Local systems on $\widetilde{\Sigma}$

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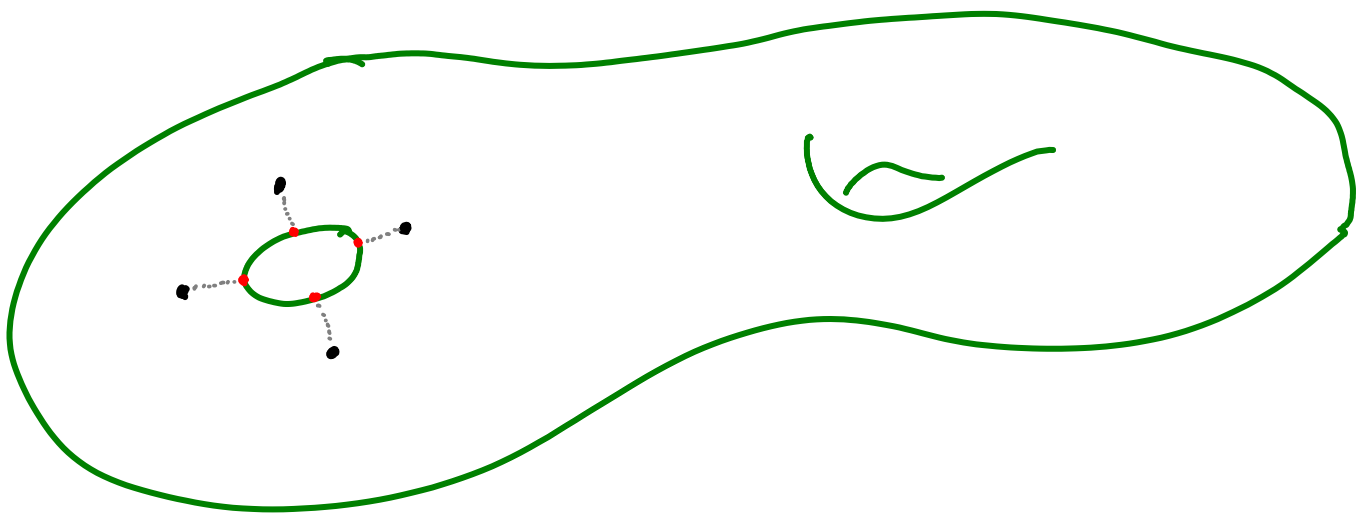

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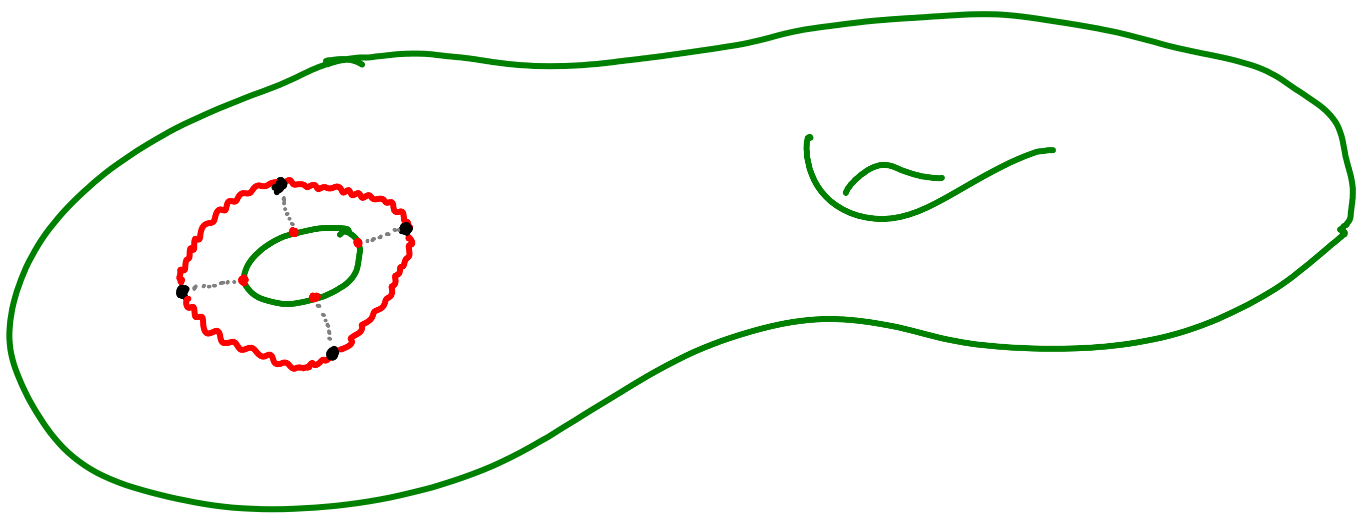

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Riemann-Hilbert
-Birkhoff

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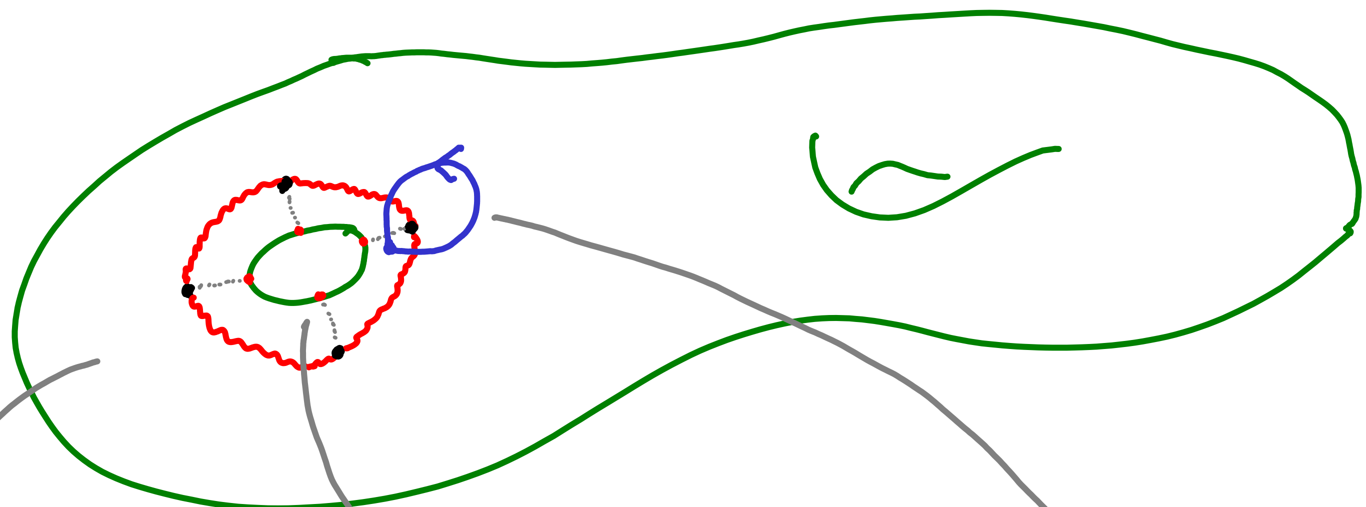

G-Local system on $\widetilde{\Sigma}$ + H_i reduction inside halos + Local monodromy in $\mathbb{S}\mathrm{to}_d$

Thm (– '11)

The space of Stokes representations $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ is a smooth affine variety and is (naturally) a quasi-Hamiltonian $\underset{\sim}{H}$-space ($\underset{\sim}{H} = H_1 \times \cdots \times H_m$)

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Corollary $\mathcal{M}_B(\Sigma, G) := \mathrm{Hom}_{\mathbb{S}}(\Pi, G) / \underset{\sim}{H}$

inherits an intrinsic Poisson structure (algebraically) with symplectic leaves $\mu^{-1}(e)/\underset{\sim}{H}$ for $e = (e_1, \ldots, e_m) \subset \underset{\sim}{H}$

[$\mathcal{M}_B$ classifies irreg. connections with the given irreg. types & Betti weights zero (else use $\hat{e}$)]

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- Have notion of irreducible Stokes representations:

Symplectic leaves of $\mathrm{Hom}^{irr}_{\mathbb{S}}(\Pi, G)/\underset{\sim}{H}$ are hyperkähler and diffeomorphic to spaces of meromorphic Higgs bundles

(Biquard–B. 2004)
$G = GL_n(\mathbb{C})$

Also studied stability for $\underset{\sim}{H} \circlearrowright \mathrm{Hom}_g(\Pi, G)$:

Hilbert-Mumford + general quasi-Hamiltonian properties $\Longrightarrow$

Thm If $\mathcal{C}$ sufficiently generic semisimple conjugacy class

then $\mu^{-1}(\mathcal{C})/\underset{\sim}{H}$ algebraic symplectic orbifold

($\underline{\text{smooth}}$ symplectic algebraic variety if $G = Gl_n(\mathbb{C})$)

Wild character varieties

(G = connected complex reductive gp)

$$\Sigma \longmapsto \mathrm{Hom}_{\mathbb{S}}(\Pi, G) / \underset{\sim}{H}$$

Irregular curve

Poisson variety

$$
\begin{tikzcd}
 & & \mathcal{M}_{Dol} \\
\Sigma \arrow[r, Rrightarrow] & \mathcal{M}(\Sigma) \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{M}_{DR} \\
 & & \mathcal{M}_{Betti}
\end{tikzcd}
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Σ: irregular curve

$\mathcal{M}(\Sigma)$: Hyperkahler manifold

$\mathcal{M}_{Dol}$: meromorphic Higgs bundles

mero. Hitchin systems (Bottacin - Markman)

$\mathcal{M}_{DR}$: mero. connections

$\mathcal{M}_{Betti}$: Stokes representations

Def^n

A family $\Sigma \to \mathbb{B}$ of irregular curves $(\Sigma_p, a_i, Q_i)_{p \in \mathbb{B}}$ is "admissible" if

1) The fibres Σ_p remain smooth

2) None of the marked points a_i coalesce

3) For each root $\alpha \in R$

$$\mathrm{PoleOrder}(\alpha \circ Q_i) \in \mathbb{Z}_{\geq 0}$$

is a constant function on $\mathbb{B}$

Thm

If $\Sigma \longrightarrow \mathbb{B}$ is an admissible family of irregular curves

$$\Sigma_p = \pi^{-1}(p), \qquad p \in \mathbb{B}$$

get algebraic Poisson action

$$\pi_1(\mathbb{B}, p) \circlearrowleft \mathrm{Hom}_{\mathbb{S}}(\Pi(p), G) / \underset{\sim}{H}$$

"The Betti moduli spaces $M_B(\Sigma_p, G)$ form a local system of (Poisson) varieties"

E.g. 1) $G = \mathbb{C}^*$ all deformations admissible

Baker functions are hoz. sections of such irreg. connections (on spectral curve)

$\Big\downarrow$ Krichever

Solutions of KdV etc

"times of KdV hierarchy $\longleftrightarrow$ irreg. deformation parameters"

$$\exp\left(x w + t_2 w^2 + t_3 w^3 + \cdots\right) \qquad \begin{cases} w = 1/z \\ x = t_1 \end{cases}$$

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$$\exp\left(xw + t_2 w^2 + t_3 w^3 + \cdots\right) \qquad \begin{cases} w = 1/z \\ x = t_1 \end{cases}$$

2) $Q = \dfrac{-A_0}{z}$, $A_0 \in \mathfrak{t}_{reg}$ $\left(dQ = \dfrac{A_0}{z^2}\, dz \right)$

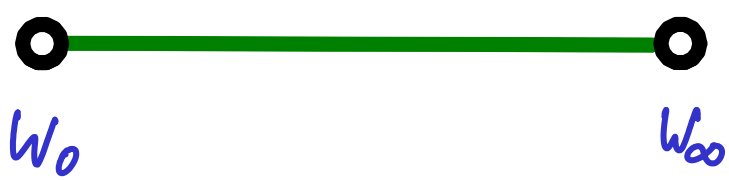
$\pi_1(\mathfrak{t}_{reg}) =$ Pure $\mathfrak{g}$-braid group

New isomonodromy systems

New isomonodromy systems

Revisit the JMMS equations (Jimbo-Miwa-Mori-Sato 1980)

Choose two finite dimensional vector spaces W_0, W_∞



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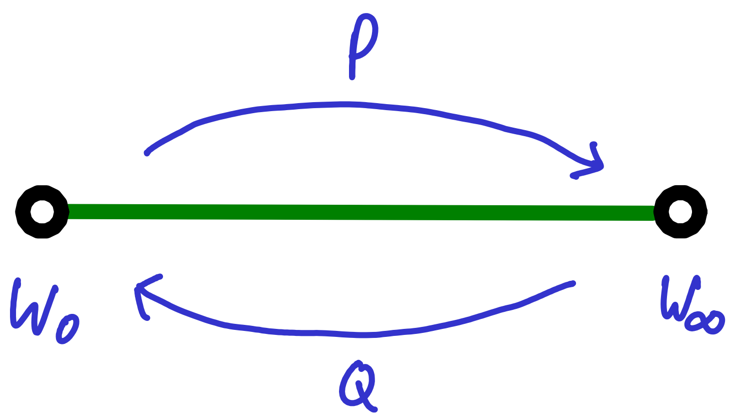

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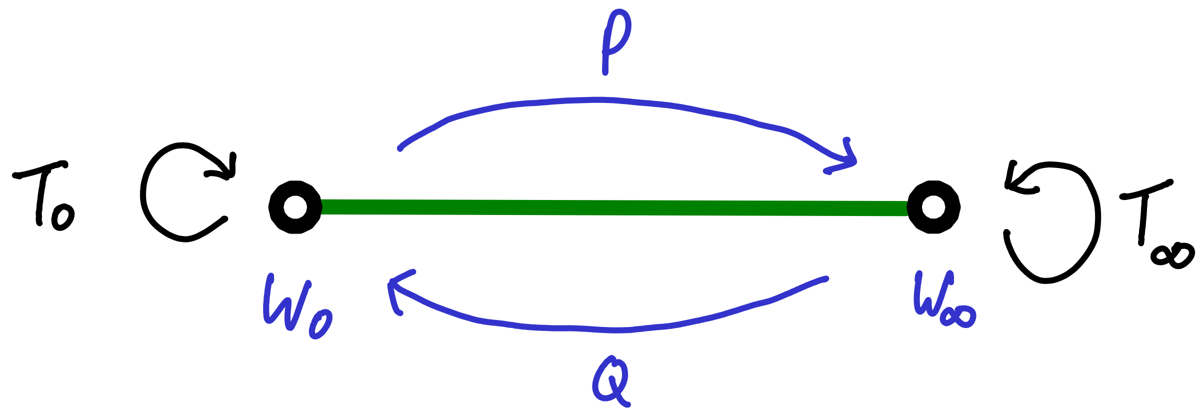
Choose two finite dimensional vector spaces W_0, W_∞

$$\begin{tikzcd} W_0 \arrow[loop left, "T_0"] \arrow[r, bend left, "P"] & W_\infty \arrow[l, bend left, "Q"] \arrow[loop right, "T_\infty"] \end{tikzcd}$$

T_i diagonalisable; Eigenvalues of T_i are the times

– no further coalescences permitted

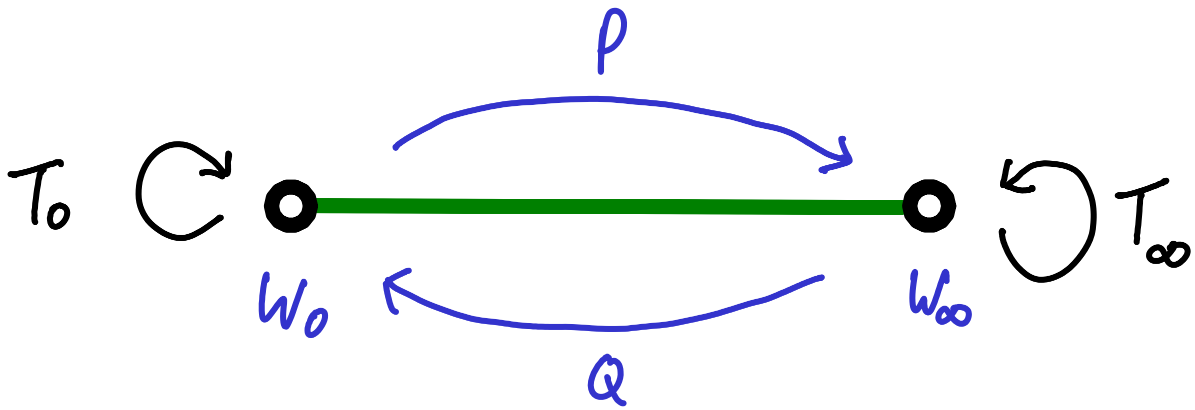
– eigenspaces fixed



Can write JMMS equations as follows:

$$dQ = Q\,\widetilde{PQ} + \widetilde{QP}\,Q + T_0\,Q\,dT_\infty + dT_0\,Q\,T_\infty$$

$$-dP = P\,\widetilde{QP} + \widetilde{PQ}\,P + T_\infty\,P\,dT_0 + dT_\infty\,P\,T_0$$



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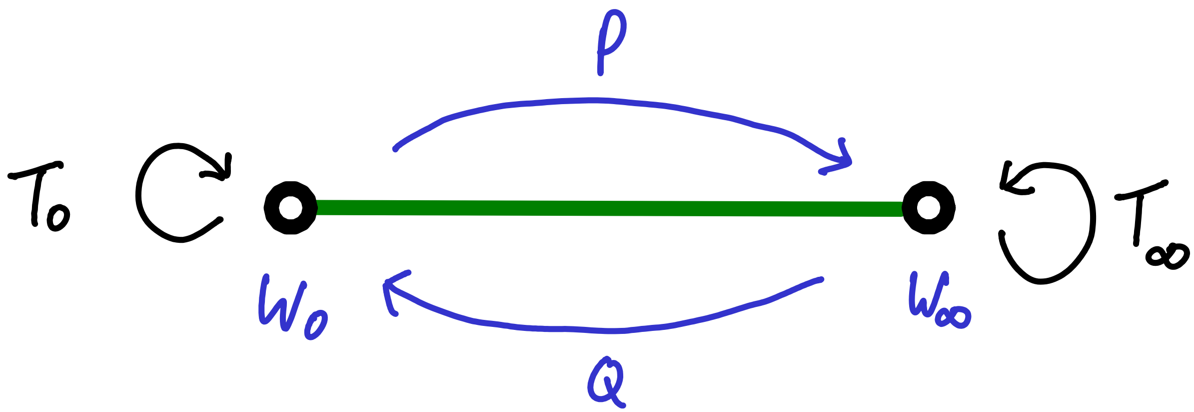
where $\tilde{R} = \mathrm{ad}_{T_i}^{-1}[dT_i, R]$ for $R \in \mathrm{End}(W_i)$

$$\left(\tilde{R}_{ab} = R_{ab}\, d\log(t_a - t_b) \quad \text{if } T_i = \sum t_a \mathbb{1}_a\right)$$



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E.g. $T_0 = 0$ JMMS $\Leftrightarrow$ Schlesinger equations

$$\begin{tikzcd} W_0 \arrow[loop left, "T_0"] \arrow[r, dash] \arrow[r, bend left=40, "P"] & W_\infty \arrow[l, bend left=40, "Q"] \arrow[loop right, "T_\infty"] \end{tikzcd}$$

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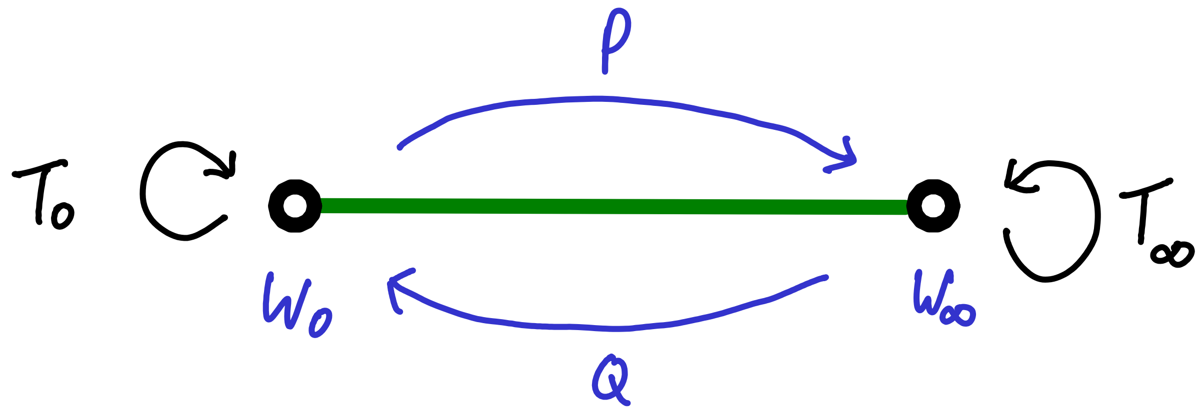
Thm (Harnad '94)

The permutation $(W_0, W_\infty, P, Q, T_0, T_\infty) \longmapsto (W_\infty, W_0, Q, -P, -T_\infty, T_0)$ preserves the JMMS equations

$$
\begin{tikzcd}
W_0 \arrow[loop left, "T_0"] \arrow[r, dash] \arrow[r, bend left=50, "P"] & W_\infty \arrow[l, bend left=50, "Q"] \arrow[loop right, "T_\infty"]
\end{tikzcd}
$$

Harnad's duality $(W_0, W_\infty, P, Q, T_0, T_\infty) \longmapsto (W_\infty, W_0, Q, -P, -T_\infty, T_0)$ basically flips over the graph.



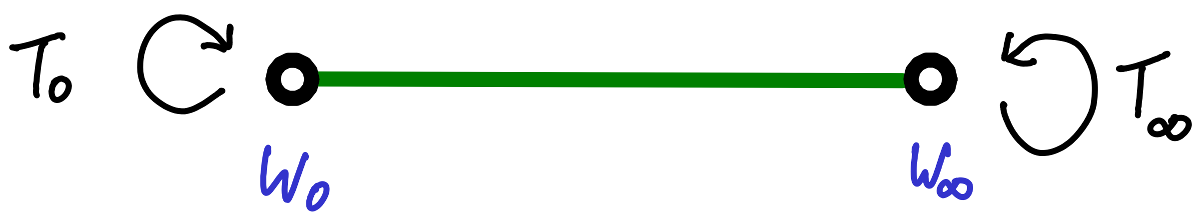
Harnad's duality $(W_0, W_\infty, P, Q, T_0, T_\infty) \longmapsto (W_\infty, W_0, Q, -P, -T_\infty, T_0)$
basically flips over the graph.

JMMS system controls isomonodromic deformations of

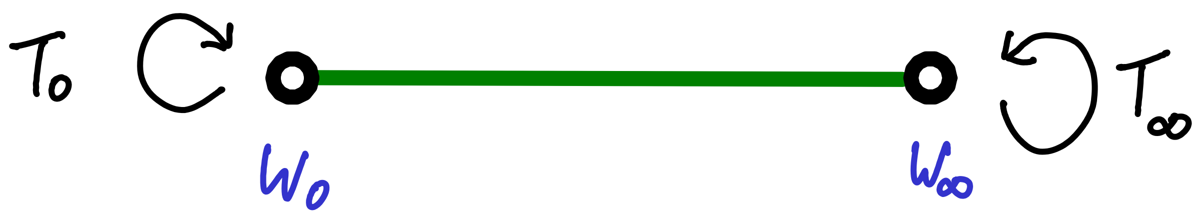
$$\left(T_0 + Q\,(z - T_\infty)^{-1} P\right) dz \qquad \text{on} \qquad W_0 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

so $W_0 \leftrightarrow W_\infty$ changes rank of the vector bundle

Splaying / additive fission



Splaying / additive fission

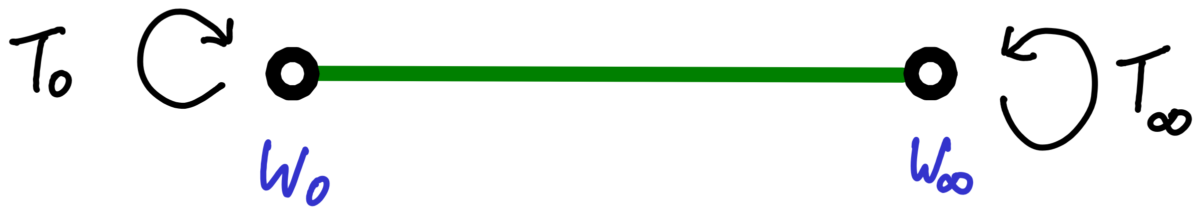


W_0, W_∞ decompose into eigenspaces of T_0, T_∞:

$$W_j = \bigoplus_{i \in I_j} V_i$$

(I_0, I_∞ label eigenspaces)

Splaying / additive fission



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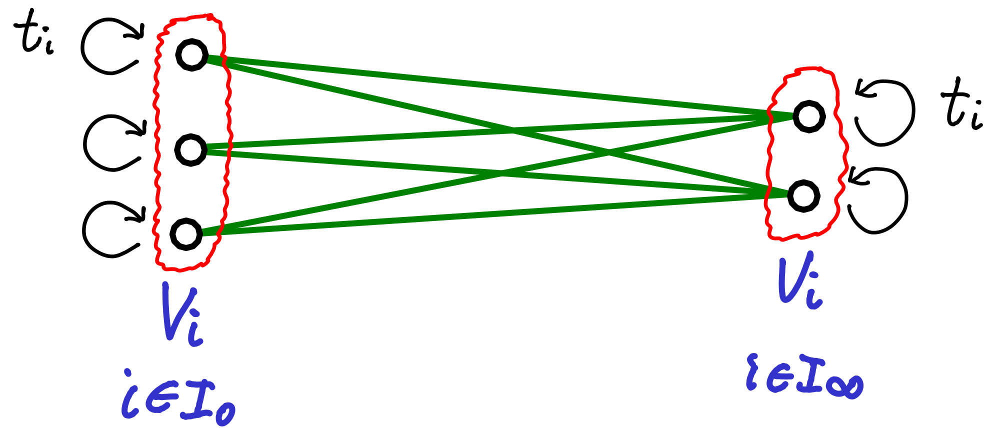
$$T_j = \sum_{i \in I_j} t_i \, \mathrm{Id}_i \qquad \begin{cases} t_i \in \mathbb{C} & \text{eigenvalues/times} \\ \mathrm{Id}_i = \mathrm{Id}_{V_i} \in \mathrm{End}(W_j) \end{cases}$$

Splaying / additive fission



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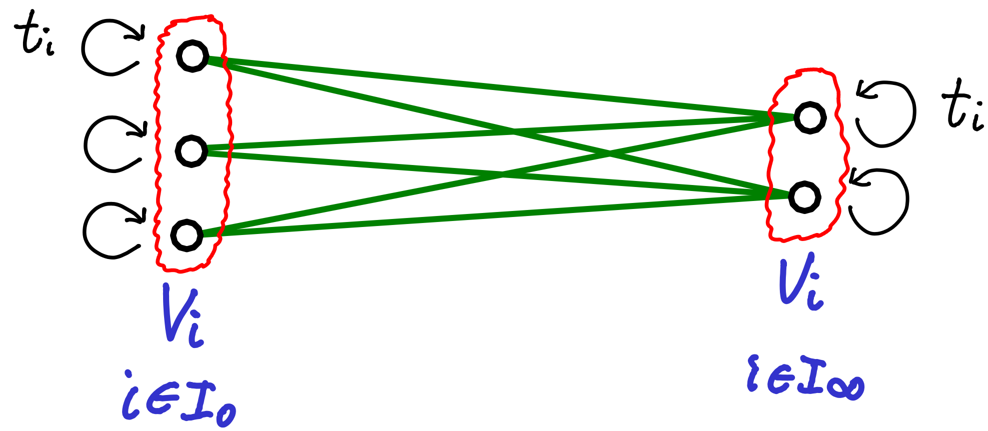


Dependent variables P, Q decompose:

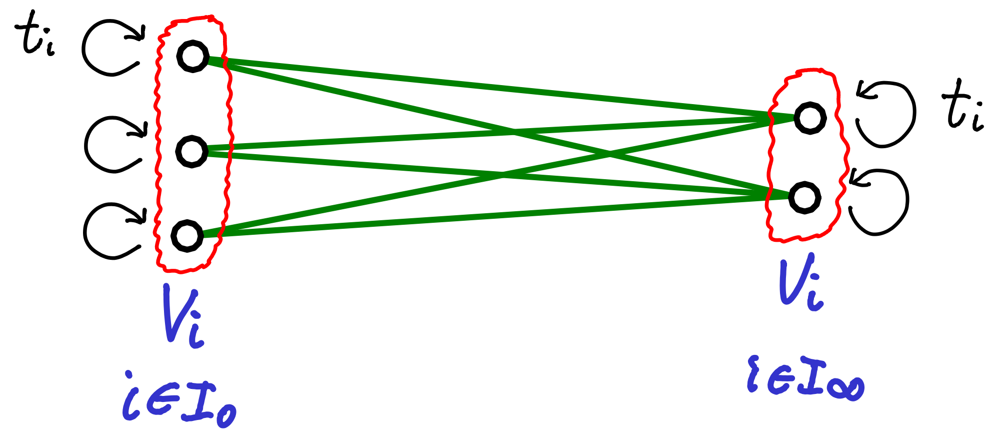
nodes
$I = I_0 \sqcup I_\infty$

$$(P, Q) \iff p_{ij} : V_j \to V_i \quad \forall i,j \in I \text{ s.t. } \exists \text{ edge } i - j$$

$$\cap$$

$$\mathrm{Hom}(W_0, W_\infty) \oplus \mathrm{Hom}(W_\infty, W_0)$$

Splaying / additive fission



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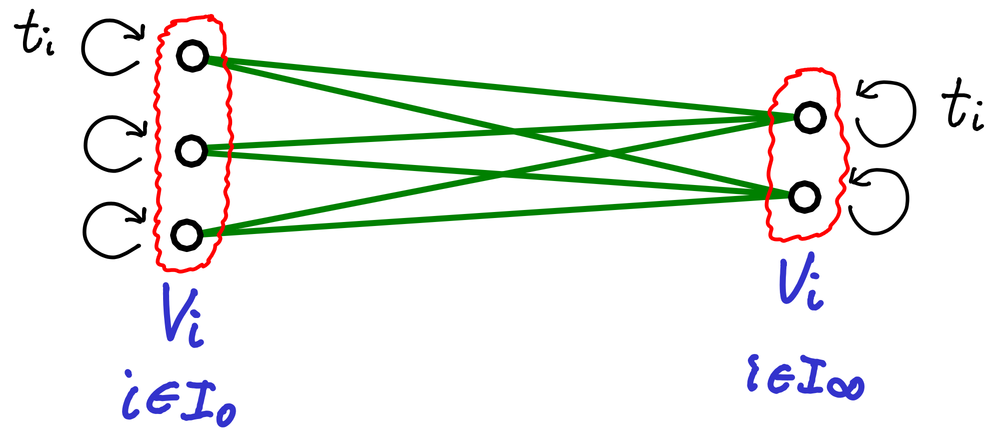
"Representation of the graph on $V = \bigoplus_{i \in I} V_i$"

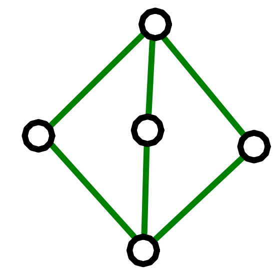
Splaying / additive fission

All complete bipartite graphs arise for the JMMS equations:

$\mathcal{G}(3,2) \cong$



$\cong$



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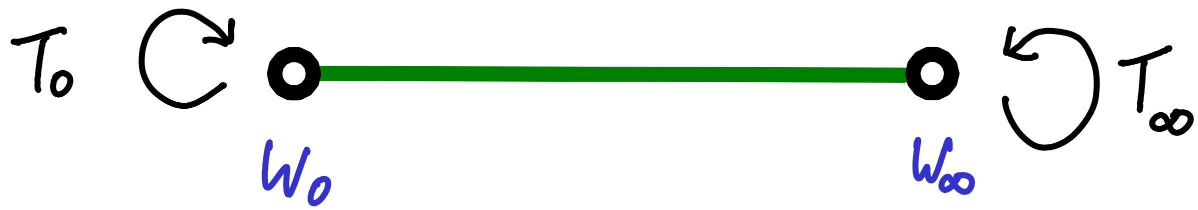
"Representation of the graph on $V = \bigoplus_{i \in I} V_i$"

Generalisation:

Replace initial graph



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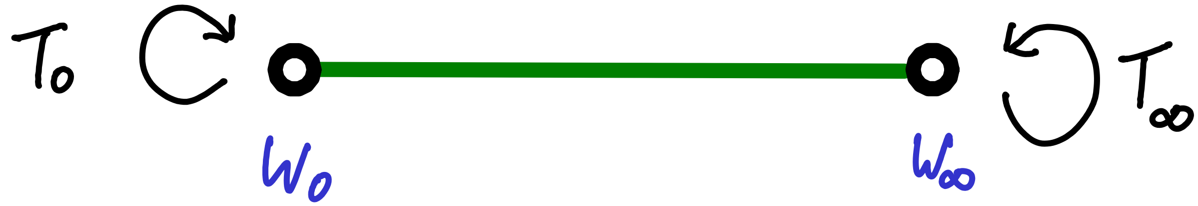
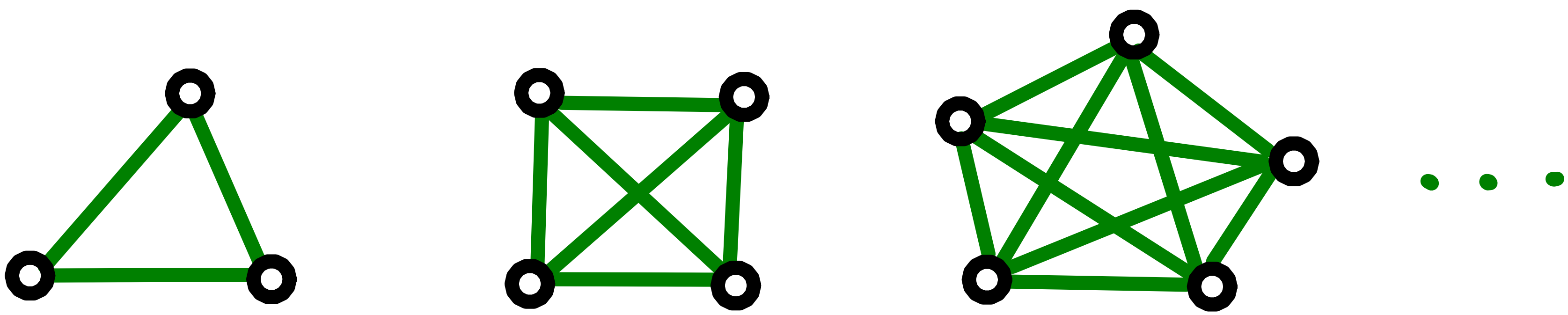


by an arbitrary complete graph:



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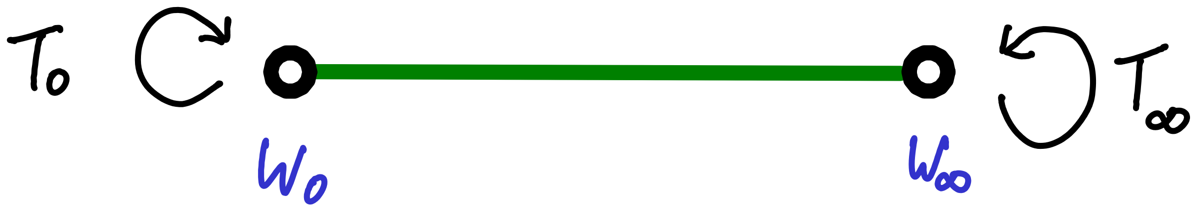

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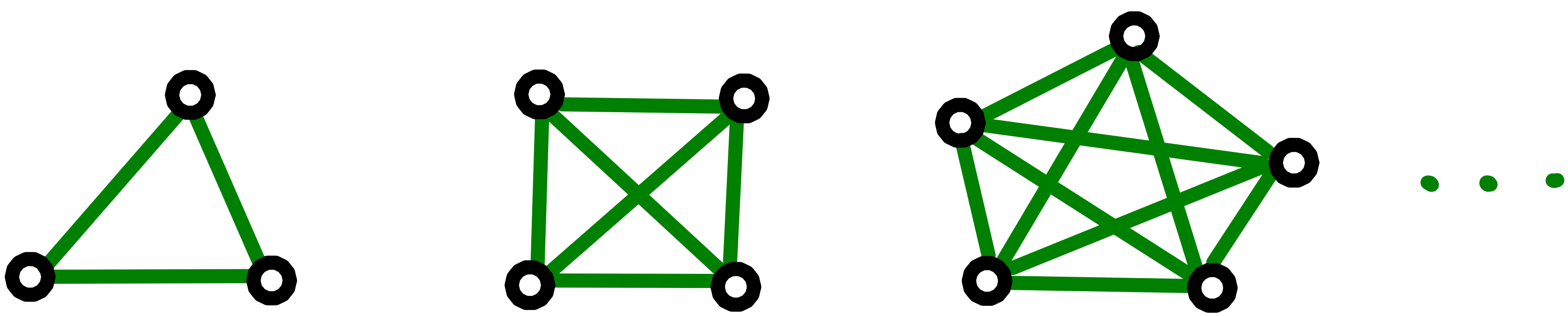

Label nodes by points $\mathcal{J} = \{a_j\} \hookrightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$

Put vector spaces W_j at nodes ($\forall j \in \mathcal{J}$), & "times" $T_j \in \mathrm{End}(W_j)$ (diagonalisable)

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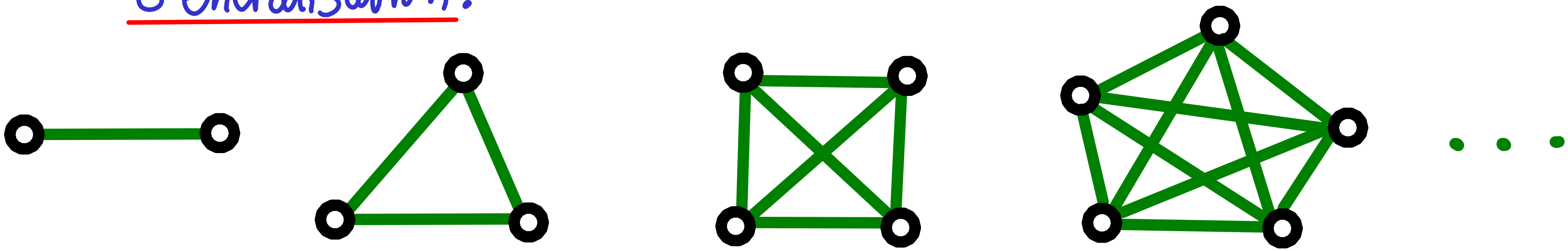

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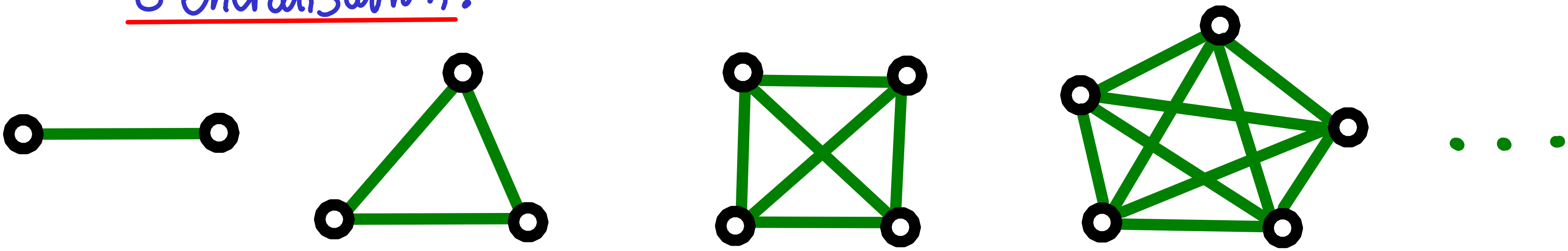

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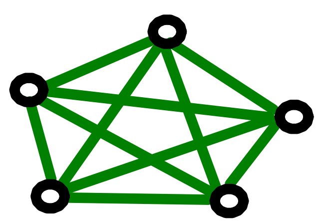
Phase space

$\mathcal{M} = \{(P,Q)\} = T^*\mathrm{Hom}(W_0, W_\infty) = \mathrm{Rep}\big($  $, W\big)$

$W = W_0 \oplus W_\infty$

$$\downarrow$$

$\mathcal{M} = \mathrm{Rep}\big($  $, W\big)$, $\quad W = \bigoplus_{j \in J} W_j$

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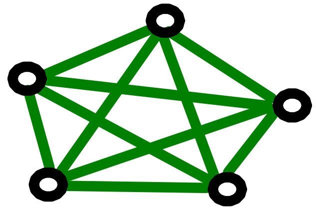
$\mathcal{M} = \mathrm{Rep}\Big($  $, W\Big)$, $\quad W = \bigoplus_{j \in J} W_j$

Point of $\mathcal{M}$ consists of maps $B_{ij} : W_j \to W_i \quad \forall i \neq j \in J$

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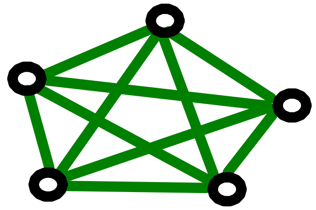
$\mathcal{M} = \mathrm{Rep}\big($  $, W\big)$, $\quad W = \bigoplus_{j \in J} W_j$

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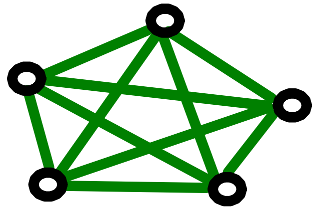
$\mathcal{M} = \mathrm{Rep}\Big($  $, W\Big)$, $W = \bigoplus_{j\in J} W_j$

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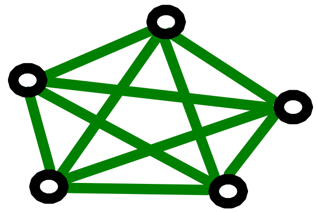
$\mathcal{M} = \mathrm{Rep}\big($  $, W\big)$, $W = \bigoplus_{j \in J} W_j$

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• Can act by Möbius transforms on $J \subset \mathbb{P}^1$ to get equiv. system

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Simply-laced isomonodromy system:

$$dB_{ij} = \sum_{k \in J} \widetilde{X_{ik} B_{ki}}\, B_{ij} + B_{ij}\, \widetilde{B_{jk} X_{kj}}$$

$$+\, dT_i\, X_{ik} B_{kj} + B_{ik} X_{kj}\, dT_j - X_{ik}\, dT_k\, X_{kj} / \phi_{ij}$$

$$+ \text{ linear terms}$$

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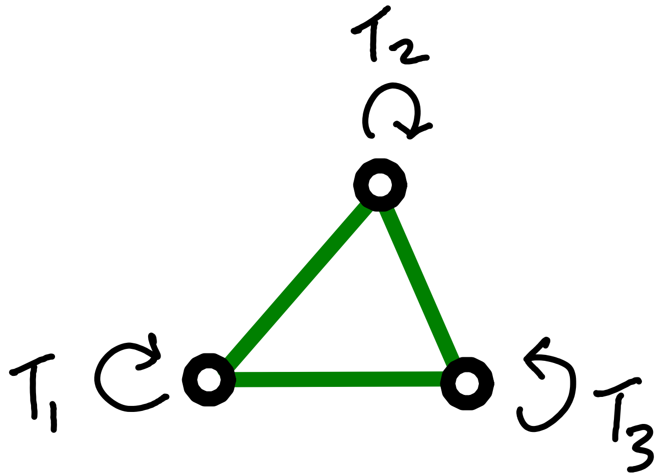
where

$$\phi_{ij} = \begin{cases} (a_i - a_j)^{-1} & \text{if } i, j \neq \infty \\ 1, -1 & j = \infty, i = \infty \text{ resp.} \end{cases}$$

$$X_{ij} = \phi_{ij} B_{ij}, \qquad (B_{ii} = 0)$$

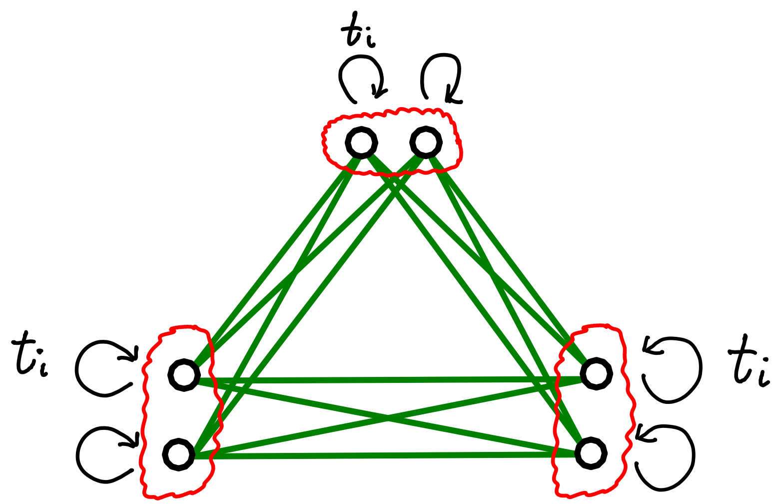
Splay/fission as before:



$$I_j = \text{Eigenspaces}(T_j)$$

$$\Downarrow$$

e.g. $|I_j| = 2 \quad \forall j$:



nodes

$$I = \bigsqcup_{j \in J} I_j$$

$$\bigoplus_{i \in I_j} V_i = W_j$$

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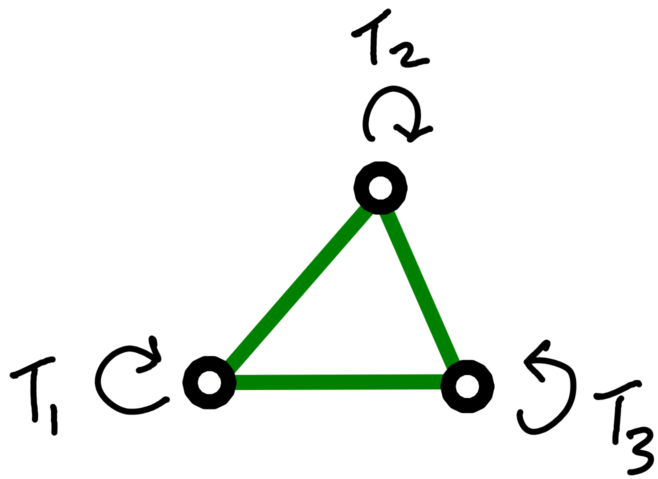

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Get all complete k-partite graphs

$k = |J| = \#\text{nodes}$

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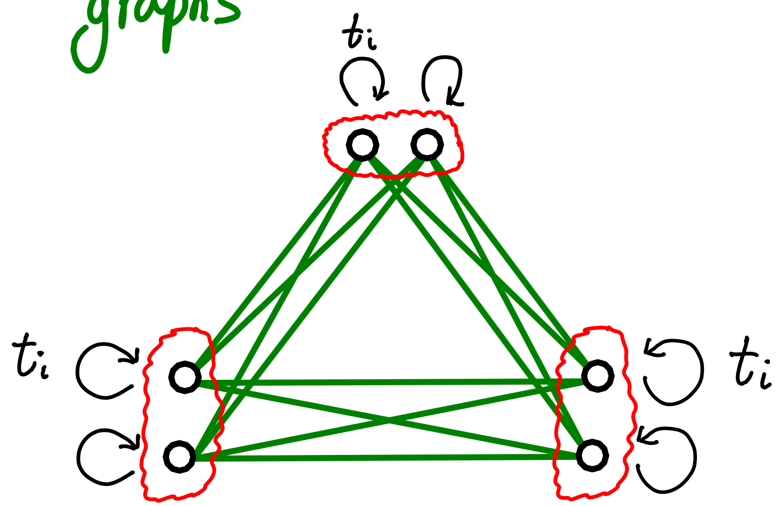
$\mathcal{G}(2,2,2) \cong$



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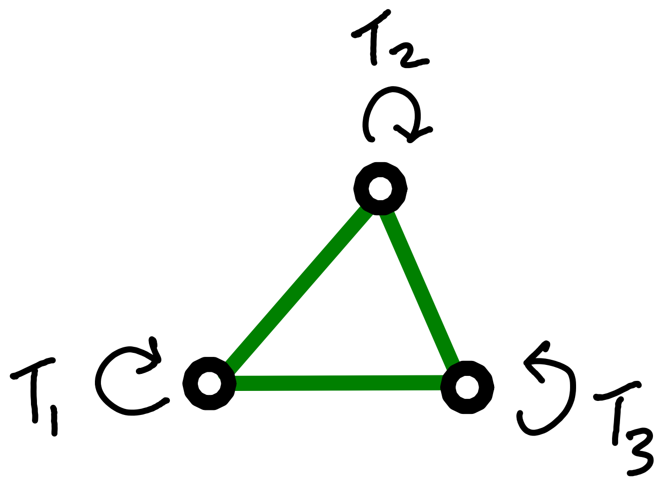

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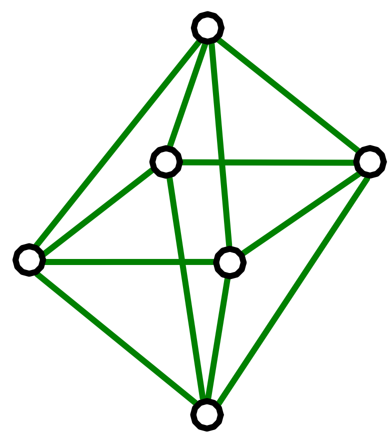

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