

Irregular connections

Dynkin diagrams

and

Fission

Philip Boalch (ENS & CNRS)

Banff 2012

Themes:

- (wild) non-abelian Hodge corresp. on curves & hyperbolic moduli spaces
- nonlinear symplectic braid group actions
- understanding such moduli spaces on \mathbb{P}^1 ($\mathcal{M}_{DR}^* \subset \mathcal{M}_{DR}$)
- symplectic geometry of wild character varieties

Wild nonabelian Hodge theory on curves

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- $G = GL_n(\mathbb{C})$, $T \subset G$
- Σ compact smooth complex algebraic curve
- $a_1, \dots, a_m \in \Sigma$ distinct points

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Definition

If $a \in \Sigma$, an irregular type Q at a is
an element $Q \in \mathfrak{t}(\hat{K}) / \mathfrak{t}(\hat{\Theta})$

If z is a local coordinate vanishing at a

$$\hat{\Theta} = \mathbb{C}[[z]], \quad \hat{K} = \mathbb{C}((z))$$

$$Q = \frac{A_r}{z^r} + \dots + \frac{A_1}{z} \quad \text{for some } A_i \in \mathfrak{t} = \text{Lie}(T)$$

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"irregular curve"

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- weights $\theta_1, \dots, \theta_m \in \mathfrak{t}_{\mathbb{R}} = X_*(T) \otimes \mathbb{R} \subset \mathfrak{t}$
($(\theta_i)_{jj} \in [0, 1)$ $j=1, \dots, n$)

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Let $\mathfrak{h}_i = C_{\mathfrak{g}}(Q_i) \subset \mathfrak{g}$ (centraliser)

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Note that $\theta \in \mathfrak{t}_{\mathbb{R}}$ determines a parabolic $\mathbb{P}_{\theta} \subset \mathfrak{g}$

$$\mathbb{P}_{\theta}(\mathfrak{g}) = \left\{ X \in \mathfrak{g} \mid \lim_{z \rightarrow 0} z^{\theta} X z^{-\theta} \text{ along any ray exists} \right\}$$

& similarly $\mathbb{P}_{\theta_i}(\mathfrak{h}_i) \subset \mathfrak{h}_i$ & \mathfrak{h}_i is Levi of $\mathbb{P}_{\theta_i}(\mathfrak{h}_i)$

Consider triples (V, ∇, γ)

- $V \rightarrow \Sigma$ rank n holom. vector bundle
- $\nabla : V \rightarrow V \otimes \Omega^1(*D)$ mero. connection $D = \sum a_i$
- $\gamma = (\gamma_i)_{i=1}^m$ flags in fibres V_{a_1}, \dots, V_{a_m}

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Near a_i V has a local trivialization in which

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for some $\lambda_i \in \mathbb{H}_i$

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- λ_i preserves \mathcal{F}_i (i.e. $\lambda_i \in \mathfrak{P}_{\theta_i}(\mathfrak{h}_i)$)
- $\pi(\lambda_i) \in \mathcal{O}_i \subset \mathfrak{l}_i$ ($\pi : \mathfrak{P}_{\theta_i}(\mathfrak{h}_i) \rightarrow \mathfrak{l}_i$)

Thm (Biquard-B. '04 building on Hitchin, Donaldson, Corlette, Simpson, Simpson, Nakajima, Subbuh, ...)

The moduli space $\mathcal{M}_{\text{DR}}(\Sigma, \underline{\theta}, \underline{Q})$

of isomorphism classes of suchmero. connections which are stable and parabolic degree zero is

- a hyperkähler manifold
- canonically diffeo. to a space ofmero. Higgs bundles
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- Higgs fields should look like $-\frac{1}{z} dQ_i + \Gamma_i \frac{dz}{z} + \text{hdom.}$ near a_i
 - same 'rotation' of the weights/eigenvalues as in Simpson 1990

Irregular curve



Hyperkahler manifold

\mathcal{M}

Irregular curve (+ weighted conjugacy classes)



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\mathcal{M}

(wild nonabelian Hodge structure)

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\mathcal{M}_{Dol}

Higgs bundles

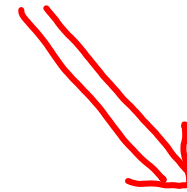
algebraic integrable systems
(mero. Hitchin systems)



\mathcal{M}_{DR}

mero. connections

isomonodromy systems



\mathcal{M}_B

monodromy & Stokes data

symplectic braid & (irregular) mapping class group actions

Braiding/Mapping class group actions

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(PB 2001)

(extended to irregular case)
Simpson 1994

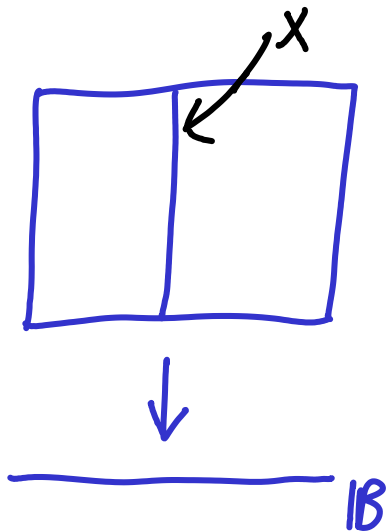
1900's (higher) Painlevé equations

~ 1980 Sato Miwa Jimbo Ueno ...

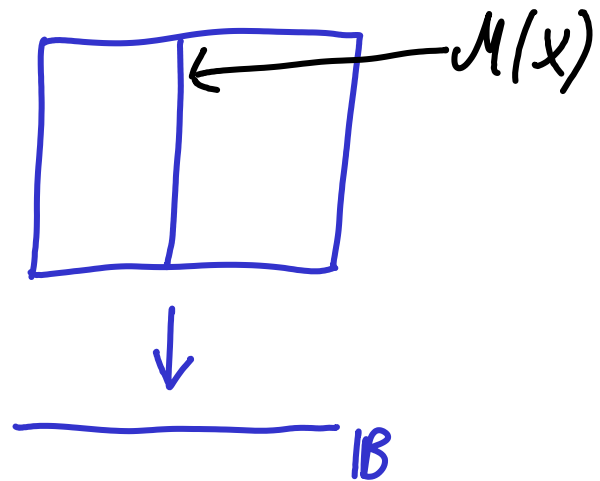
Braiding / Mapping class group actions

“(isomonodromy = Nonabelian Gauss-Manin connection)”
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Reg. case:



family of curves with marked points



‘family’ of moduli spaces
- nonlinear fibre bundle with flat algebraic connection
- Betti spaces form a local system of varieties

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Look at admissible deformations of irreg. curve, such that:

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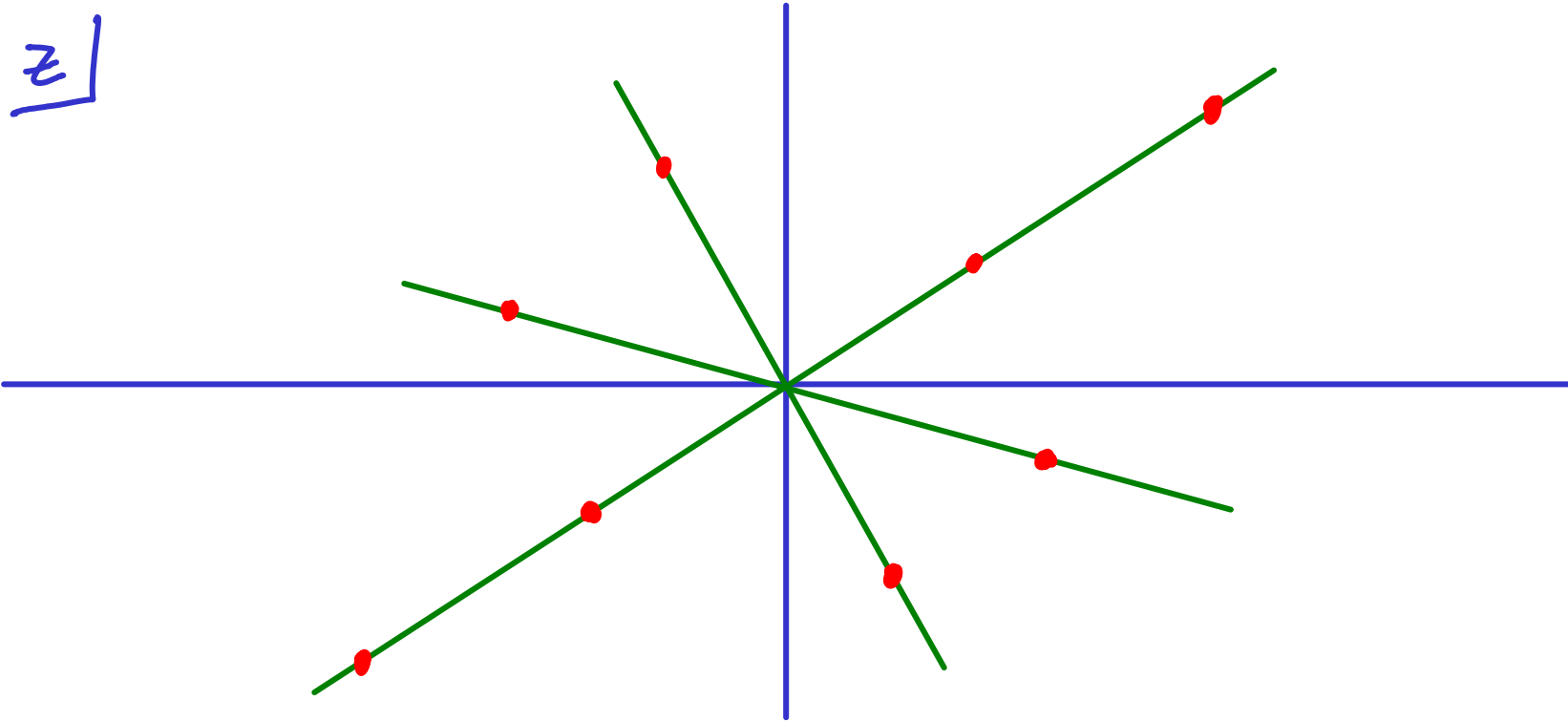
Then again the Betti spaces form a local system of varieties

and get notion of isomonodromic deformations of mera. connections

(f. Jimbo-Miwa-Ueno '81 (GL_n), PB '02, '11 (other G , $A_r \notin t_{\text{reg}}$)

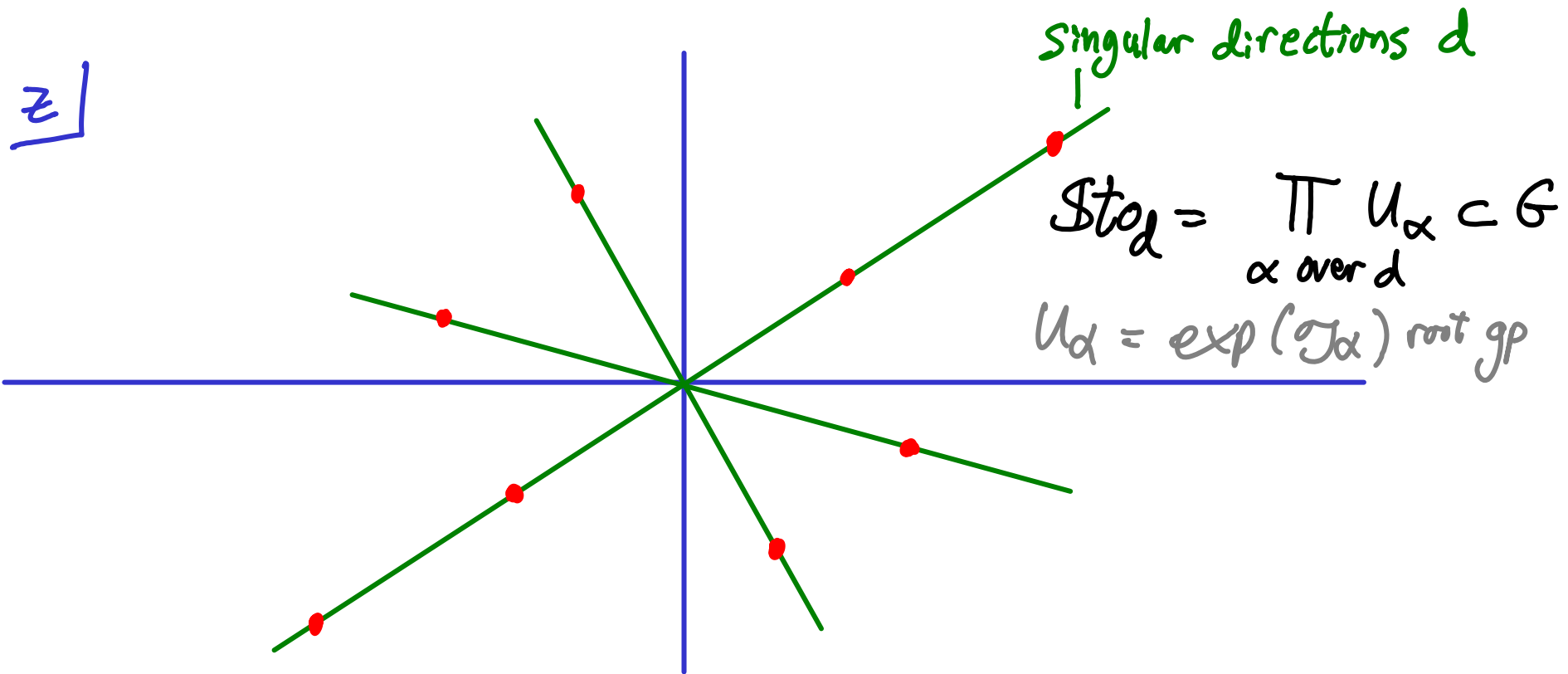
Simplest example (PB '02) $r=1$, $Q = \frac{-A_1}{z}$, $A_1 \in \mathbb{T}_{\text{reg}}$

Plot roots on z -plane: $\langle A_1, \mathbb{R} \rangle \subset \mathbb{C}^*$



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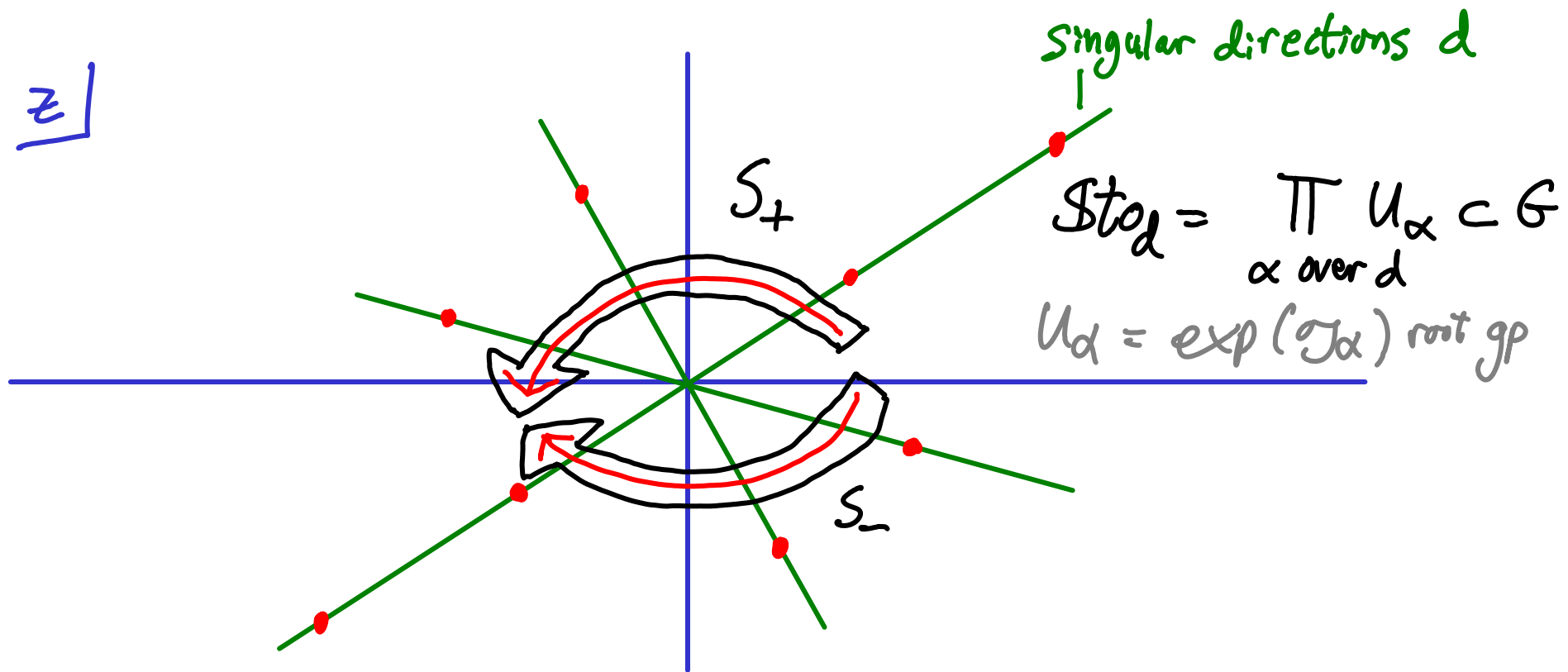
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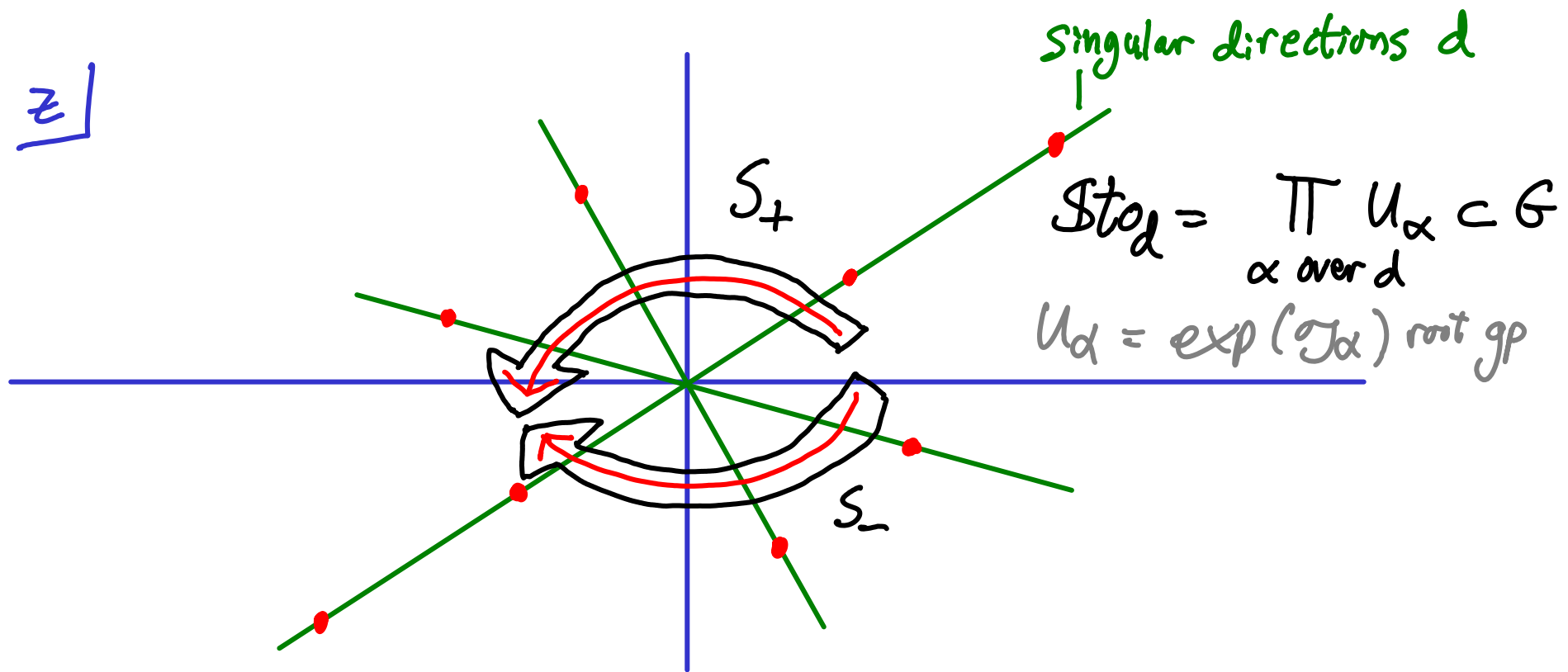


$$\{\text{Stokes data}\} = \prod_d Stod \cong U_+ \times U_- \ni (S_+, S_-)$$

unipotent radicals of opposite Borels

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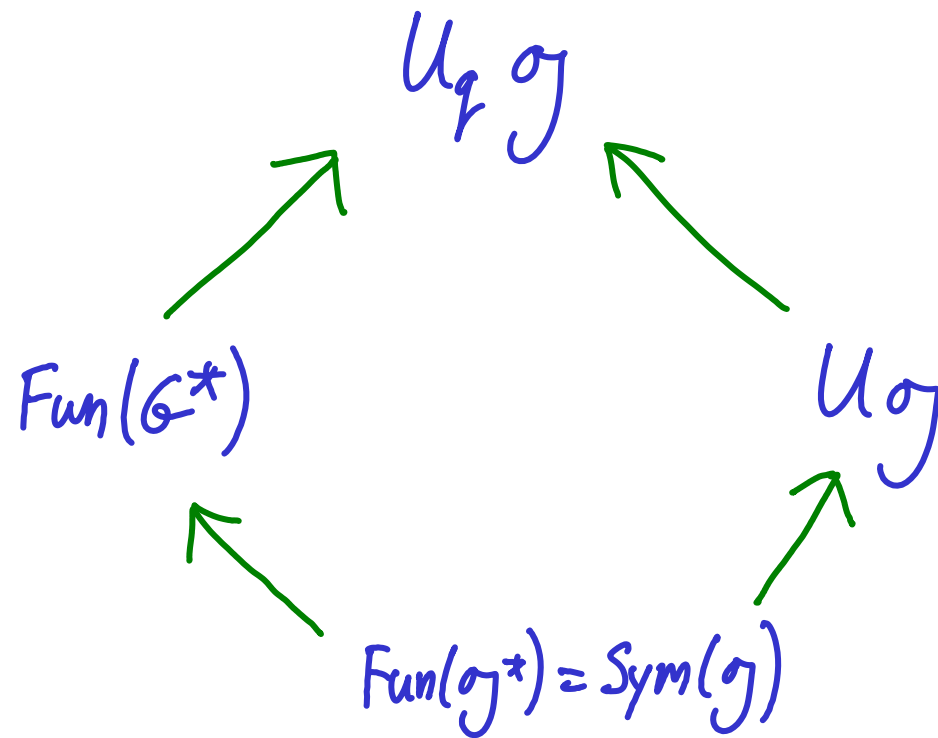
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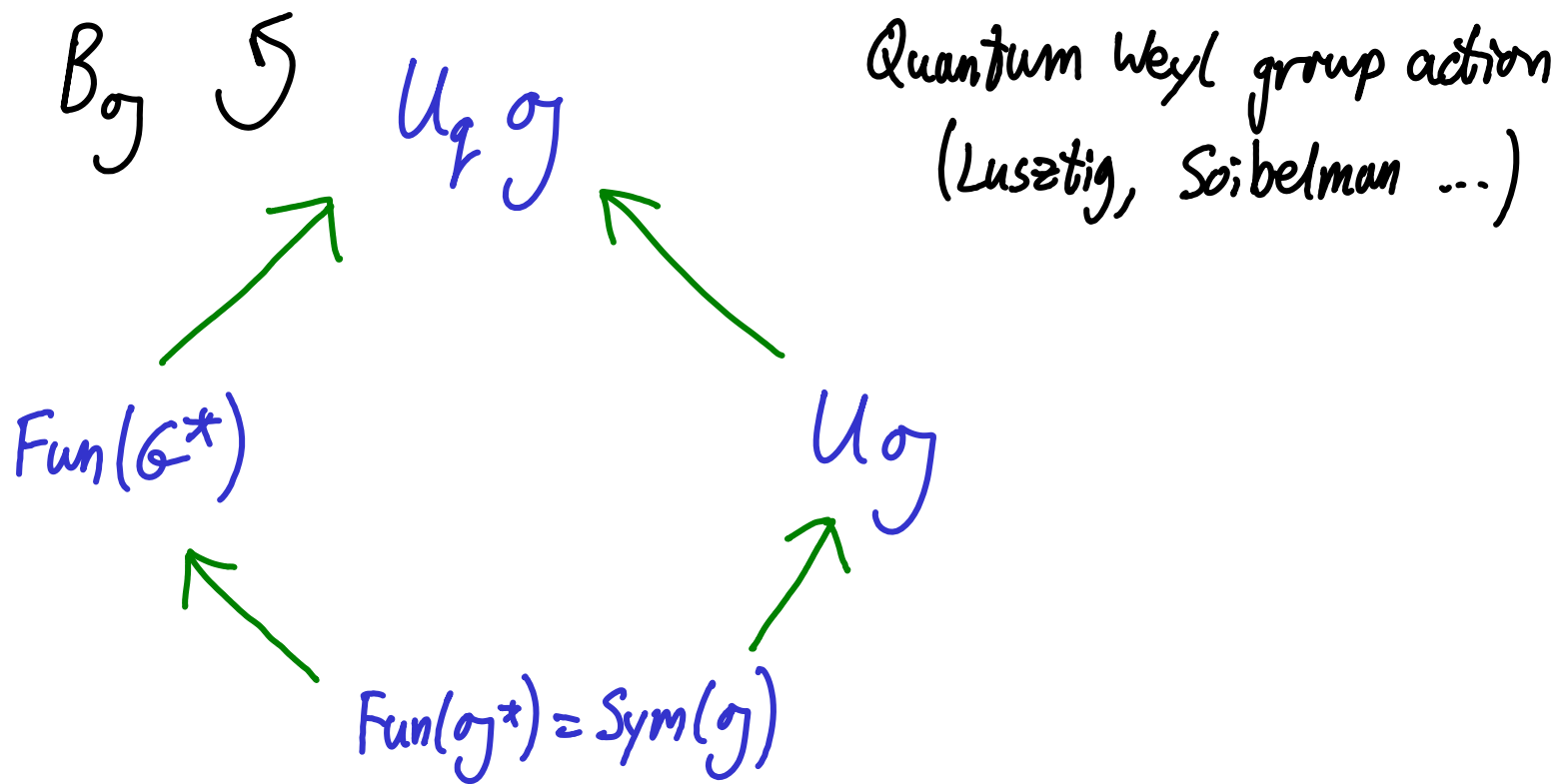
Isomonodromy: Vary $A_1 \in \mathfrak{t}_{\text{reg}}$ & keep S_{\pm} const. (locally)

In this example the resulting braided gp action had been previously seen:



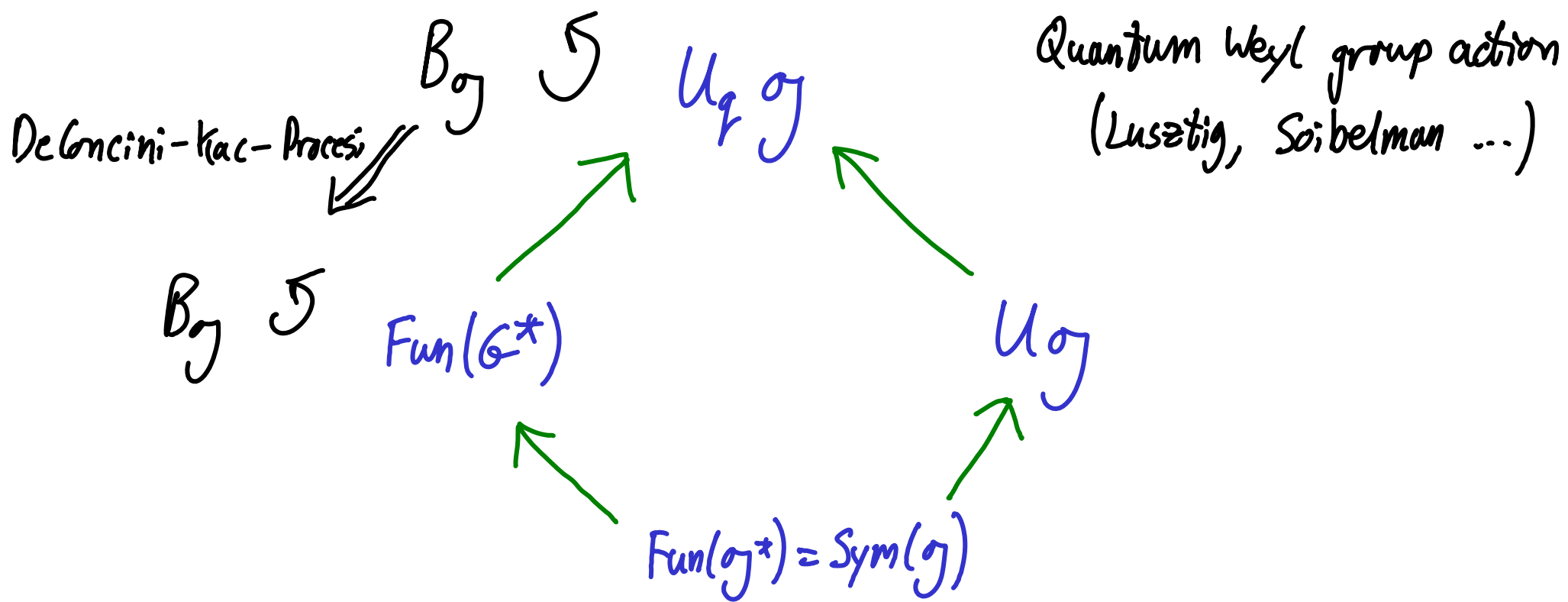
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Thm (-'02)

The DKP action arises from isomonodromy ($U_+ \times U_- =$ Stokes data)

- Purely geometric origin (not just explicit generators)
- $U_q g$ thus quantizes a moduli space of meromorphic connections

Example (cont.)

$$\mathcal{B} \in \mathfrak{g}^*$$

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Given
 $A_1 \in \mathcal{T}_{\text{reg}}$

$$\left(\frac{A_1}{z^2} + \frac{B}{z} \right) dz$$

Example (cont.)

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Given
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Example (cont.)

$$B \in \mathcal{G}^* \xrightarrow{\nu_{A_1}} \mathcal{G}^*$$

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Thm (PB '01-'02)

ν_{A_1} is Poisson & is generically a local analytic isomorphism

(\Rightarrow new direct proofs certain results of Duistermaat, Ginzburg-Weinstein, Kostant)

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Isomonodromy equations: $dB = [B, \text{ad}_{A_1}^{-1} [dA_1, B]]$

($\mathfrak{t}_{\text{reg}} = \{A_1\} = \text{'times'}$)

Formula for (part of) ν_{A_1} by Bridgeland-Toledano ~ 2008

General case is similar, space of Stokes data more complicated:

$$\{\text{Stokes data}\} = \prod_{d \in A} \mathcal{S}to_d$$

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$$\{\text{Stokes data}\} = \prod_{d \in A} \text{Sto}_d$$

- singular directions $A = \left\{ d \in S^1 \mid \begin{array}{l} e^{\alpha \circ Q(z)} \text{ has max. decay} \\ \text{as } z \rightarrow 0 \text{ along } d \text{ for some root } \alpha \end{array} \right\}$
⊗
- $\text{Sto}_d = \prod_{\alpha \in \mathcal{R}(d)} \exp(\sigma_\alpha) < G$ (unipotent subgroup)
- $\mathcal{R}(d) = \{\text{roots s.t. } \otimes \text{ holds in direction } d\} \subset \mathcal{R}$

Guide to moduli spaces on \mathbb{P}^1

Typically

$M^* \subset M$
└
open part where
bundle holom. trivial / \mathbb{P}^1

& M^* again a complete hyperkahler manifold
"approximation" to more transcendental
metric on M

Classical hyperkahler mfd's

① Complex coadjoint orbits $\theta \in \mathfrak{g}^*$

(Kronheimer, Biquard, Koralev)

If pole divisor $2(0) + (\infty) \subset \mathbb{P}^1$

have examples where

$$\mathcal{M}^* \cong \theta //_{\lambda} T_K$$

$$\left[\mathcal{M}_{\text{Betti}} = \mathcal{L} //_{\lambda} T, \quad \mathcal{L} \subset \mathfrak{G}^* \text{ symplectic leaf} \right]$$

($T_K \subset T$ compact torus)

② T^*G (Kronheimer)

If pole divisor $2(d) + 2(\infty) \subset \mathbb{P}^1$

have examples where

$$\mathcal{M}^* \cong T_K \amalg_{\lambda_1} T^*G \amalg_{\lambda_2} T_K$$

$$\left[\begin{array}{l} \mathcal{M}_{\text{Betti}} = T \amalg_{\lambda_1} D \amalg_{\lambda_2} T \\ D \subset (G \times G^*)^2 \quad \text{Lu-Weinstein double sympl. groupoid} \end{array} \right]$$

③ ALE spaces deformations of \mathbb{C}^2/Γ

(Eguchi-Hanson, Gibbons-Hawking, Hitchin, Kronheimer)

$\dim_{\mathbb{R}} = 4$ (gravitational instantons / quaternionic curves)

$\Gamma \subset SU_2$ finite \leftrightarrow ALE affine Dynkin graph

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Fact

In cases $E_8, E_7, E_6, D_4, A_3, A_2, A_1$

have \mathcal{M} s.t. $\mathcal{M}^* \subset \mathcal{M}$ is corresponding ALE space

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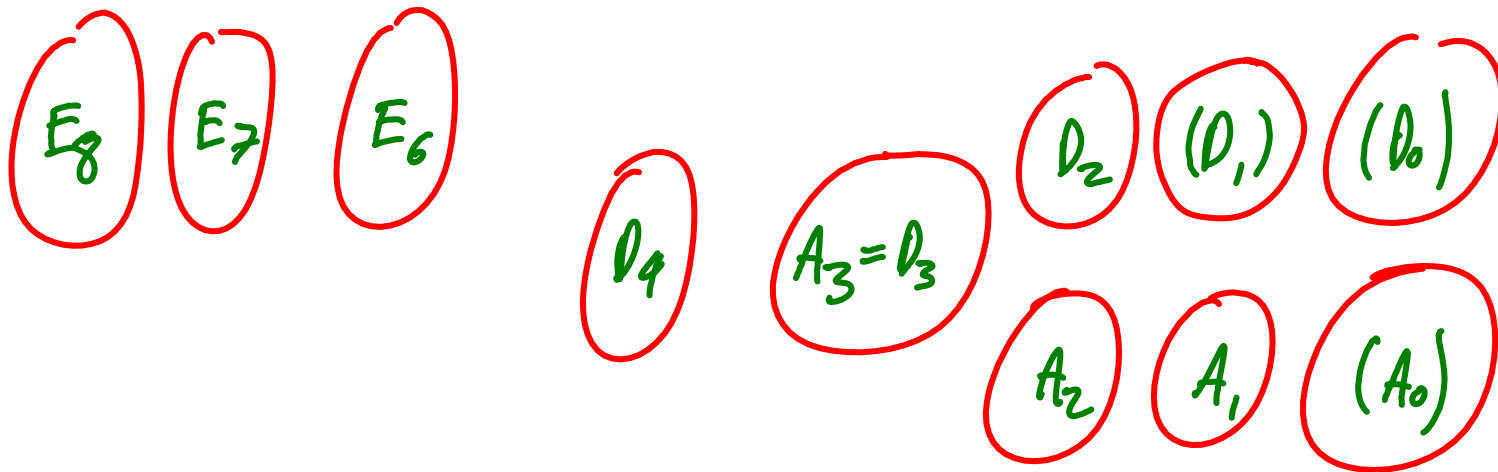
$\Gamma \subset SU_2$ finite \leftrightarrow ALE affine Dynkin graph

Fact In cases E_8, E_7, E_6, D_4 (logarithmic realisations), A_3, A_2, A_1 (only irregular realisations) have \mathcal{M} s.t. $\mathcal{M}^* \subset \mathcal{M}$ is corresponding ALE space

	Pole orders
A_3	2 + 1 + 1
A_2	3 + 1
A_1	4

- Okamoto found in 1987 the corresponding affine Weyl groups are the sym gps of the corresponding Painlevé equations

Rough classification (of \mathcal{M}_s) in $\dim_{\mathbb{C}} = 2$:



reg \leftarrow $\left|$ \rightarrow irreg

④ (Nakajima) Quiver varieties



$\text{Hom}(V, W) \oplus \text{Hom}(W, V)$ is hyperkahler $U(V) \times U(W)$ space

Graph = ADE dynkin graph \Rightarrow ALE space (Kronheimer)

else in general get higher dimⁿ hyperkahler mfd (or empty)

-lets consider simply-laced cases

E.g. Fuchsian case $G = \mathrm{GL}_n(\mathbb{C})$

$$M^* \cong \mathcal{O}_1 \times \dots \times \mathcal{O}_m // G$$

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point of $\mathcal{M}^* \sim$ Fuchsian system $\sum_1^m \frac{A_i}{z - q_i} dz$ $A_i \in \mathcal{O}_i$
 $\sum A_i = 0$

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Relation to quivers (Kraft-Prcesi, Nakajima, ...)

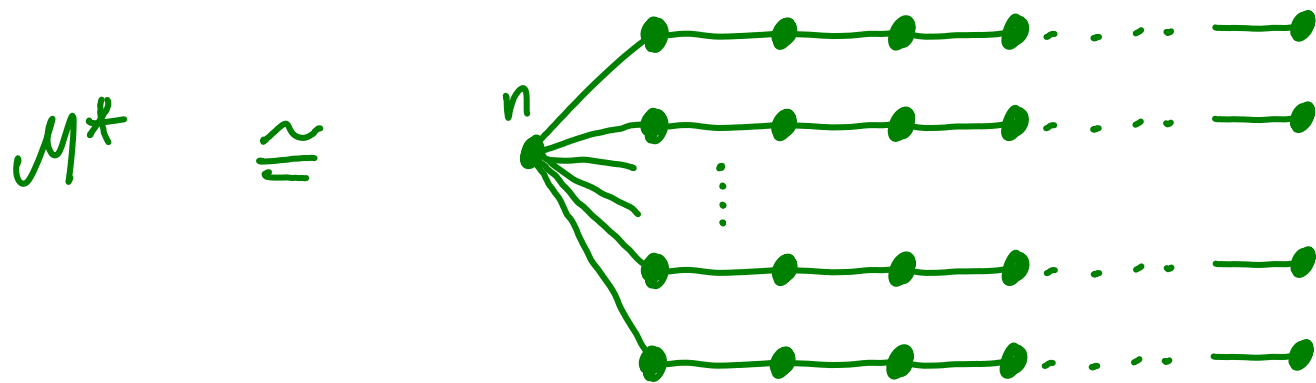
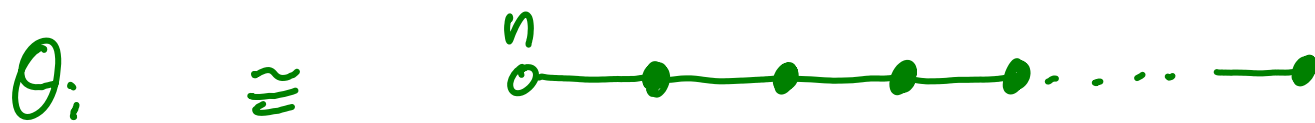
$$\mathcal{O}_i \cong \overset{n}{\circ} \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \dots \text{---} \bullet$$

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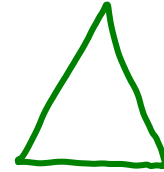
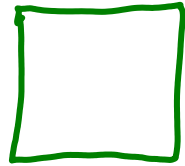
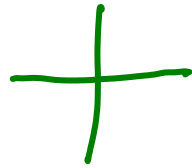
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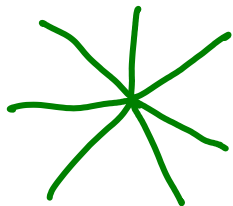
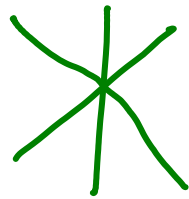
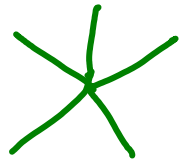
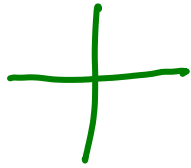


"Starshaped" quivers used by Crawley-Boevey in Deligne-Simpson problem

Recall Okamoto showed the Painlevé equations 4, 5, 6 have affine Weyl group symmetries of type A_2, A_3, D_4 resp.



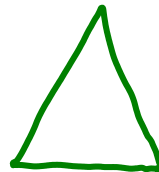
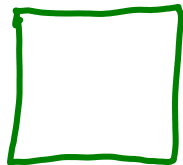
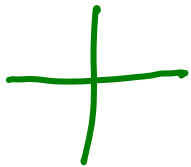
Recall Crawley-Boevey related moduli spaces of Fuchsian systems
to star-shaped quivers (building on Kraft-Procesi, Nakajima, ...)



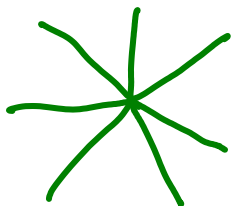
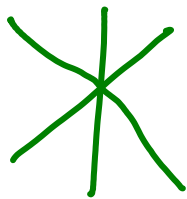
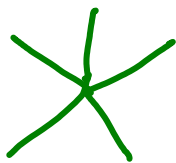
Fuchsian

Irregular

$\dim \mathcal{M} = 2$



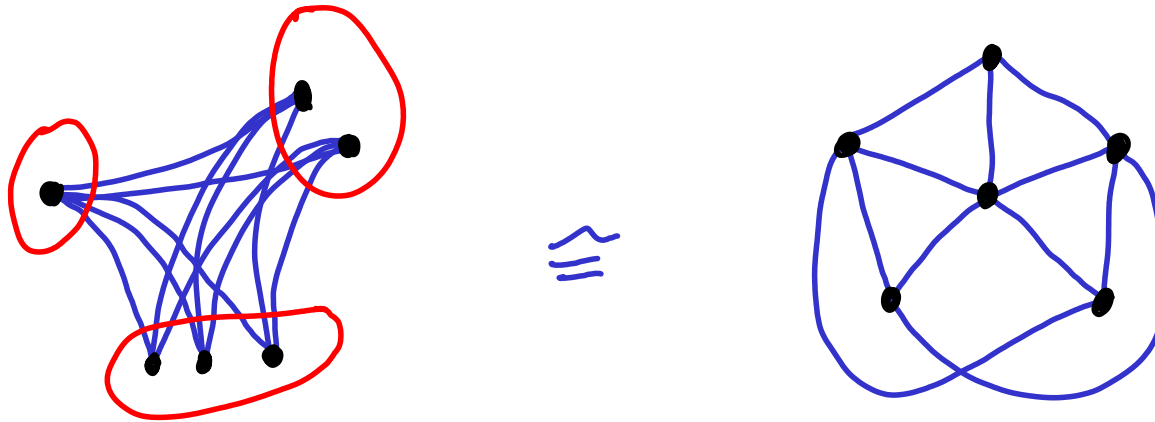
$\dim \mathcal{M} > 2$



Thm

Can take any complete k -partite graph (for any k)

E.g.



$$\Gamma(3, 2, 1)$$

- gets action of corresponding (not necessarily affine)
Kac-Moody Weyl group

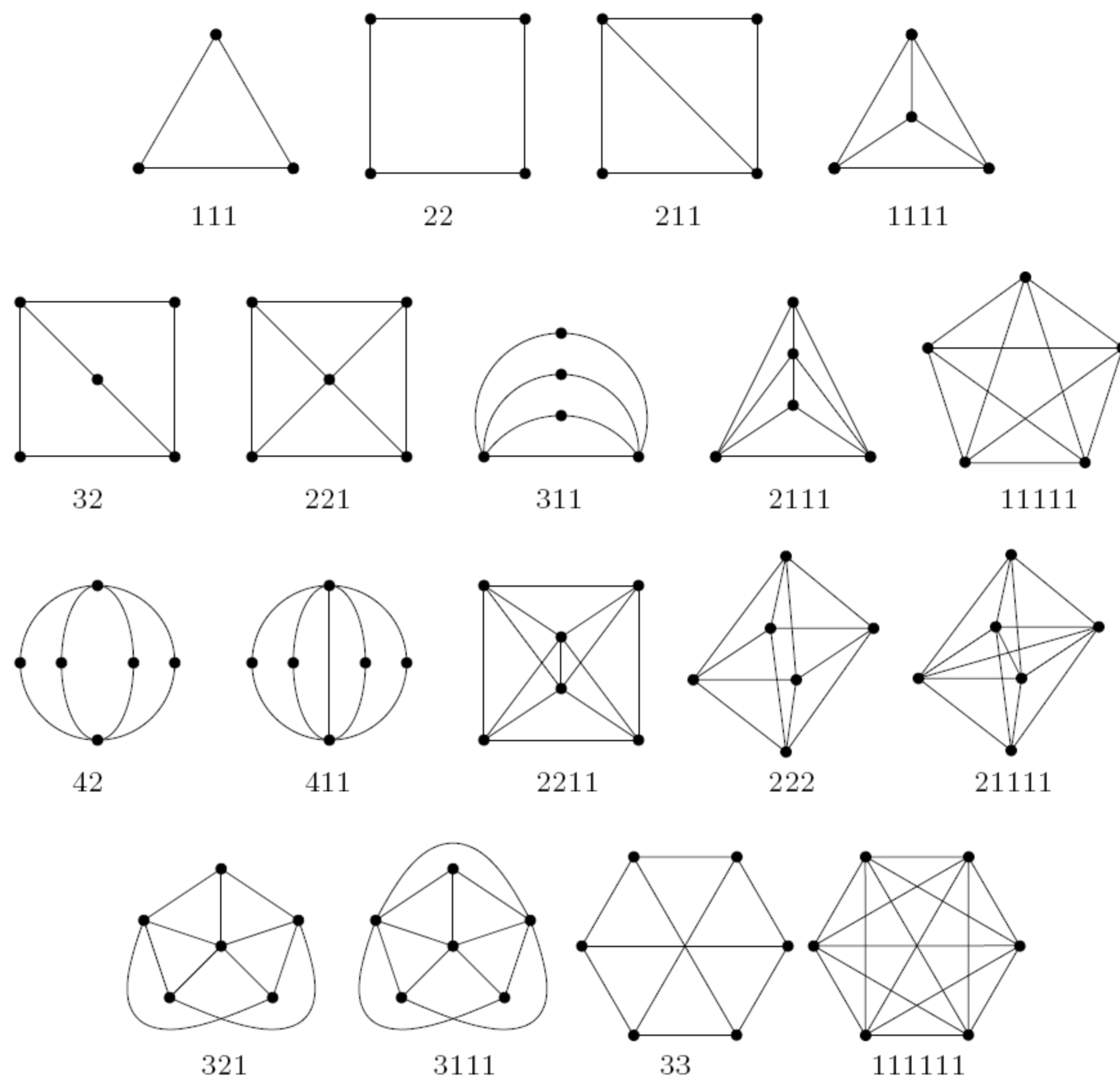


FIGURE 1. Graphs from partitions of $N \leq 6$
 (omitting the stars $\Gamma(n, 1)$ and the totally disconnected graphs $\Gamma(n)$)

More general and precise statement:

Definition A graph Γ is a

- "nonabelian Hodge graph" if there is some (rational) irregular curve Σ

s.t.

$$\begin{array}{c} \mathcal{M}^*(\Sigma) \cong \text{a quiver variety attached to } \Gamma \\ \uparrow \\ \mathcal{M}(\Sigma) \end{array}$$

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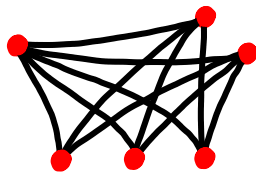
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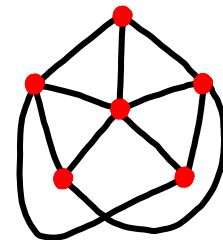
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\cong



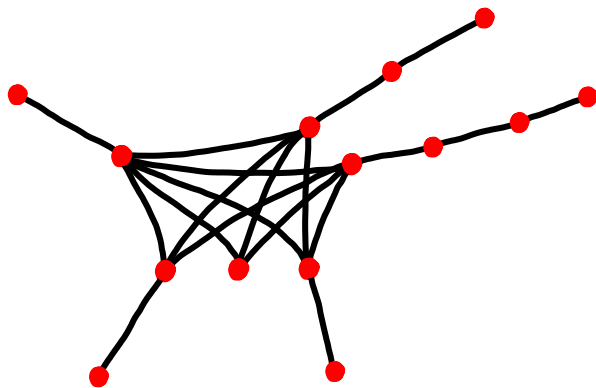
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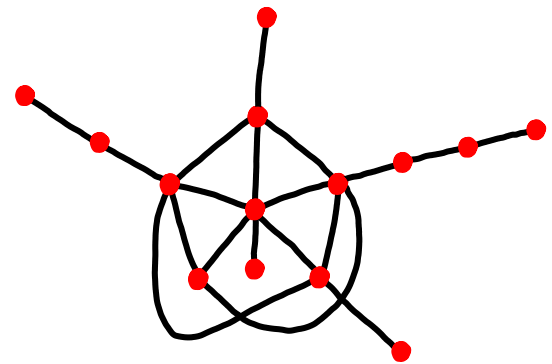
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\cong



— generalising the star-shaped graphs

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Any supernova graph is a nonabelian Hodge graph

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so can attach nonabelian Hodge structure \mathcal{M} to any such graph

& thus • a Hitchin system

• an isomonodromy system

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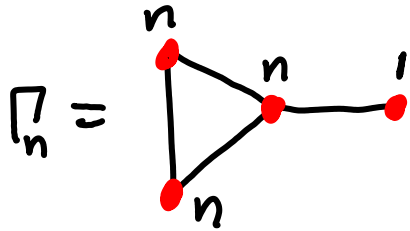
so can attach nonabelian Hodge structure \mathcal{M} to any such graph

& thus • a Hitchin system

• an isomonodromy system

Moreover Γ determines a (symmetric) Kac-Moody root system & Weyl group, and Weyl group elements lift to give isomorphisms between such systems

E.g. Higher/hyperbolic/Hilbert Parteré systems



$hP_{IV}^n := \mathcal{M}(\Gamma_n)$ dimension $2n$

E.g. Higher/hyperbolic/Hilbert Poincaré systems

$$\Gamma_n \cong \begin{array}{c} n \\ \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \\ \backslash \quad / \\ n \end{array} \text{---} 1 \quad \Rightarrow \quad hP_{IV}^n := \mathcal{M}(\Gamma_n) \quad \text{dimension } 2n$$

$$n=1 \quad hP_{IV}^1 \cong P_{IV} \quad \text{dim } 2$$

$$\mathcal{M}^*(\Gamma_n) \cong \underset{\text{diffeo}}{\text{Hilb}^n(\mathcal{M}^*(\Gamma_1))}$$

E.g. Higher/hyperbolic/Hilbert Poincaré systems

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\(\downarrow\) diffeo

Question: $\mathcal{M}(\Gamma_n) \stackrel{?}{\cong} \text{Hilb}^n(\mathcal{M}(\Gamma_1))$ (for generic parameters)

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↓
diffeo

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Similarly for any 2d Hitchin system e.g:

$$\Gamma_n = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \text{---} \bullet \text{---} \bullet \quad \Rightarrow \quad hP_V^n := \mathcal{M}(\Gamma_n) \quad \text{dimension } 2n$$

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Complex character varieties

($G =$ connected complex reductive gp)

Σ



$\text{Hom}(\pi_1(\Sigma), G) / G$

Riemann surface

Poisson variety

Quasi-Hamiltonian approach

Say $\partial\Sigma = \partial_1 \cup \dots \cup \partial_m$ ($\partial_i \cong S^1$)

Choose basepoints $b_i \in \partial_i$

Let $\Pi = \Pi_1(\Sigma, \{b_1, \dots, b_m\})$

Thm (Alekseev et al) $\text{Hom}(\Pi, G)$ is a smooth affine variety
which is naturally a quasi-Hamiltonian G^m -space

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& symplectic leaves are $\mu^{-1}(e) / \mathcal{G}^m$ ($e = (e_1, \dots, e_m) \in \mathcal{G}^m$)

Irregular Betti spaces

Let Σ be an irreg. curve (marked points a_1, \dots, a_m , irreg. types Q_1, \dots, Q_m)

Let $\hat{\Sigma} \rightarrow \Sigma$ be real oriented blow up of Σ at a_i :

(each a_i replaced by a circle ∂_i , so $\partial \hat{\Sigma} = \partial_1 \cup \dots \cup \partial_m$)

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(each a_i replaced by a circle ∂_i , so $\partial \hat{\Sigma} = \partial_1 \cup \dots \cup \partial_m$)

Then each Q_i determines:

1) A connected complex reductive group $H_i \subset G$

2) A finite set $A_i \subset \partial_i$ of singular directions at a_i

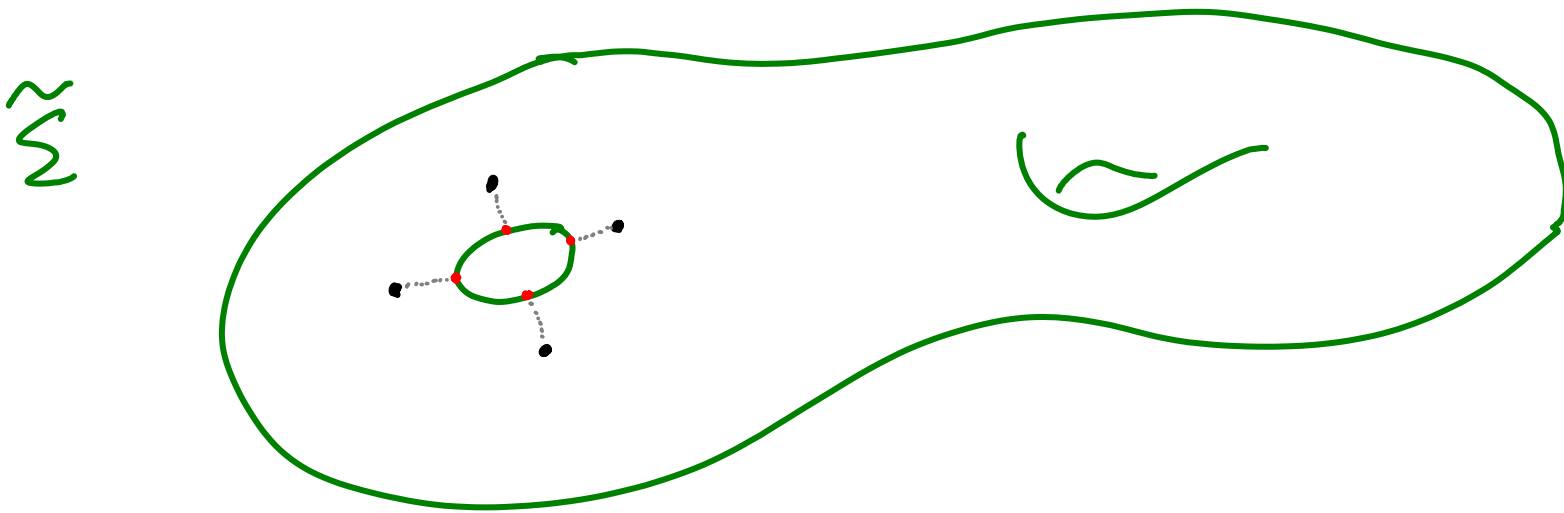
and for each $d \in A_i$

3) A unipotent group $\text{St}_d(Q_i) \subset G$ normalised by H_i

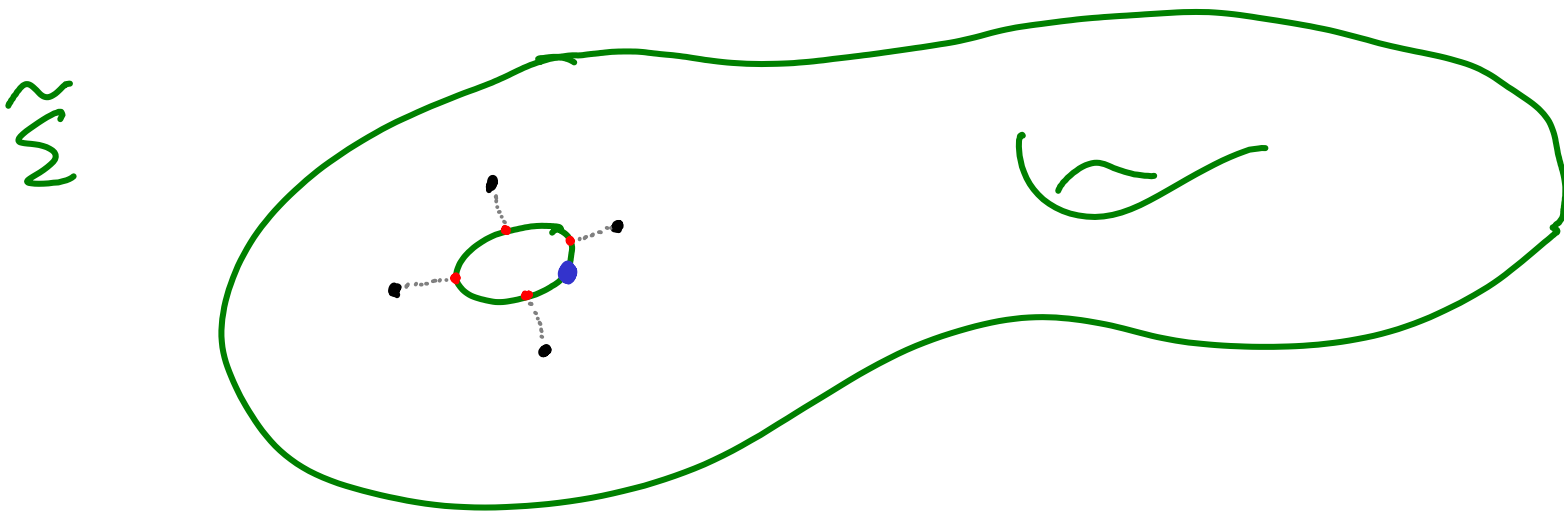
Now puncture $\hat{\Sigma}$ once in its interior near each singular

direction $d \in A_i$, $i=1, \dots, m$

and let $\tilde{\Sigma} \subset \hat{\Sigma}$ be resulting punctured surface

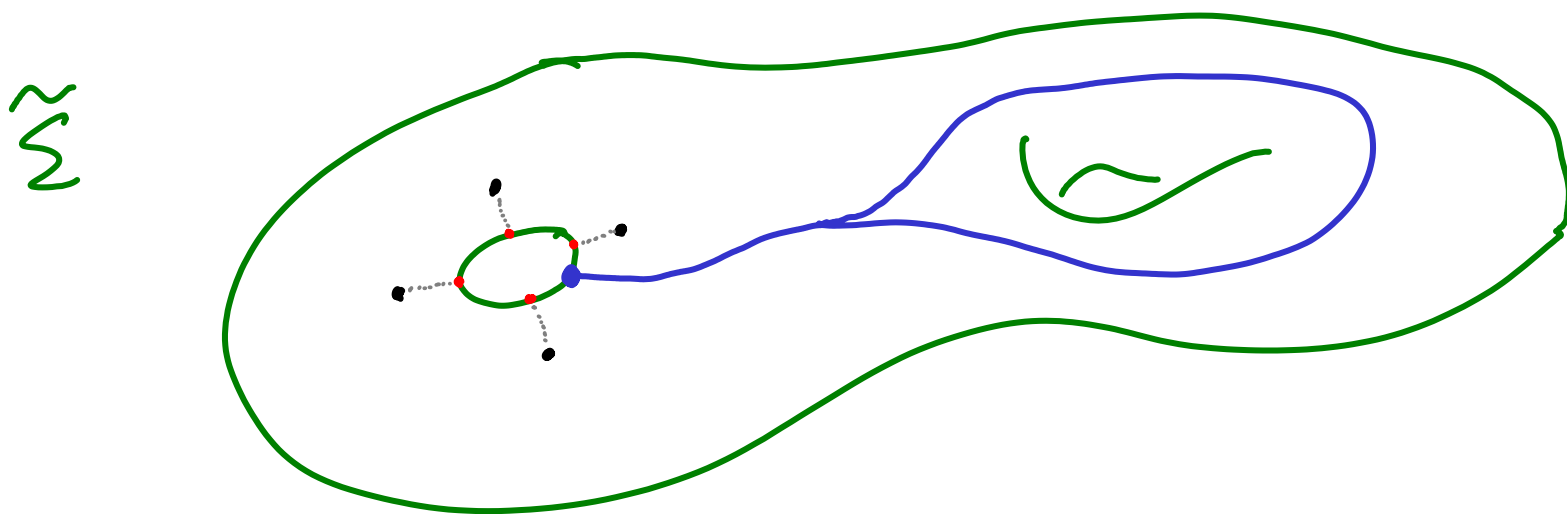


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Let $\Pi = \Pi_1(\tilde{\Sigma}, \{b_1, \dots, b_m\})$

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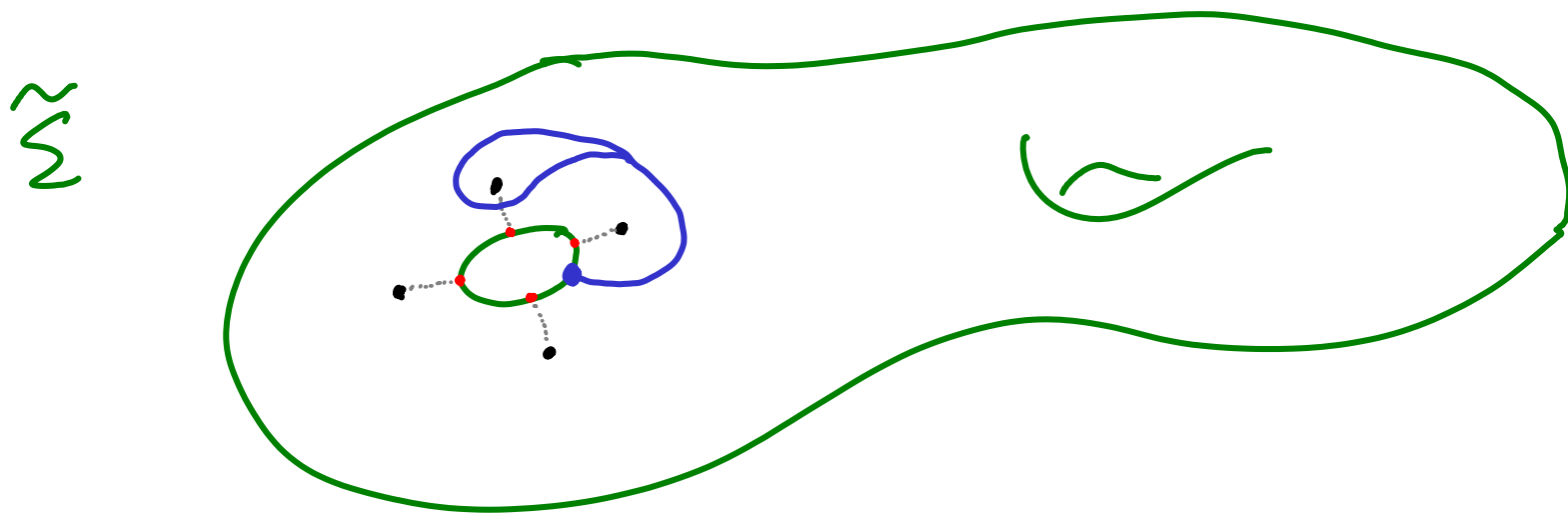


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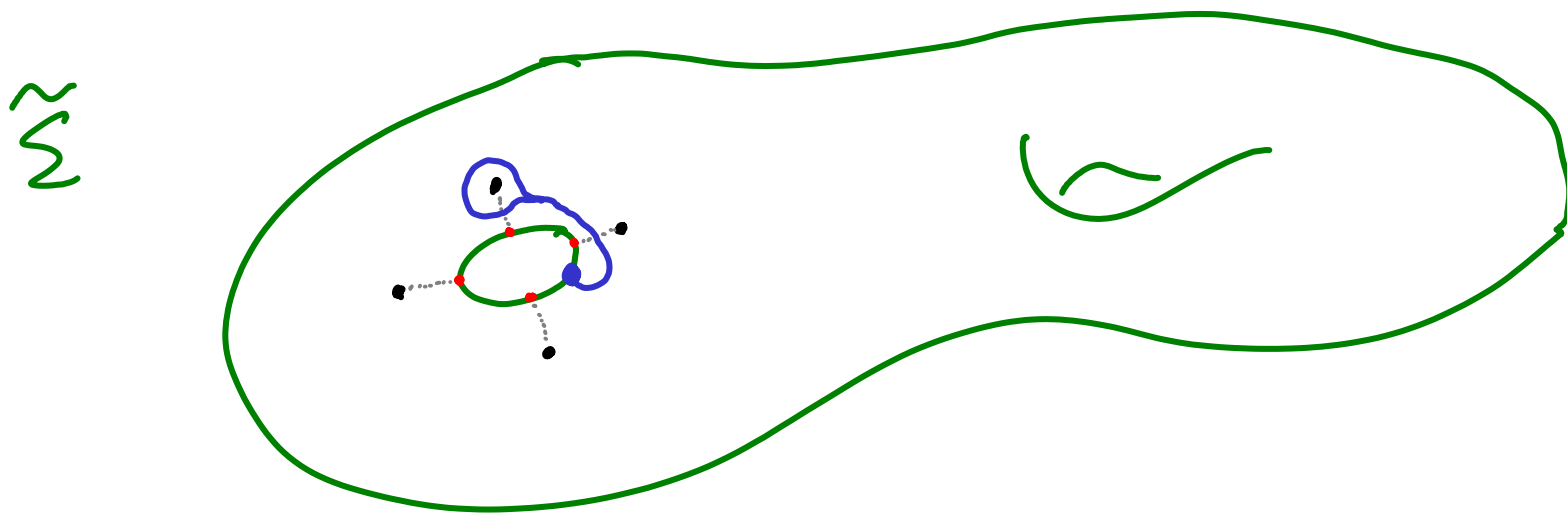
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Now consider $\text{Hom}(\Pi, G)$

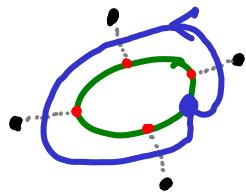
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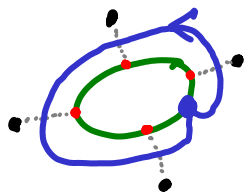
1) If $\gamma = \partial_i$ then $\rho(\gamma) \in H_i$



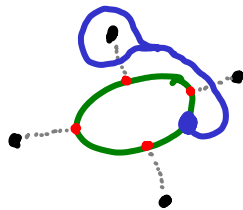
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and the subset $\text{Hom}_S^U(\pi, G)$ of "Stokes representations"
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1) If $\gamma = \partial_i$ then $\rho(\gamma) \in H_i$



2) If γ goes around ∂_i from b_i until $d \in A_i$ then loops around the corresponding puncture before returning to b_i , then $\rho(\gamma) \in St_d$



Thm

The space of Stokes representations $\text{Hom}_{\mathcal{G}}(\Pi, \mathcal{G})$ is a smooth affine variety and is (naturally) a quasi-Hamiltonian \underline{H} -space ($\underline{H} = H_1 \times \dots \times H_m$)

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Corollary

$$M_B(\Sigma, \mathcal{G}) := \text{Hom}_{\mathcal{S}}(\Pi, \mathcal{G}) / \underline{H}$$

inherits an intrinsic Poisson structure (algebraically) with

symplectic leaves $\mu^{-1}(e) / \underline{H}$ for $e = (e_1, \dots, e_m) \in \underline{H}$

M_B classifies irreg. connections with the given irreg. types
& Betti weights zero (else use \hat{e})

Wild character varieties

($G =$ connected complex reductive gp)

 Σ  $\text{Hom}_S(\Pi, G) / \underline{H}$

Irregular curve

Poisson variety

Thm

If $\Sigma \rightarrow IB$ is an admissible family of irregular curves

$$\Sigma_p = \pi^{-1}(p), \quad p \in IB$$

get algebraic Poisson action

$$\pi_1(IB, p) \curvearrowright \text{Hom}_S(\pi_1(p), G) / \underline{H}$$

"The Betti moduli spaces $M_B(\Sigma_p, G)$ form a local system of (Poisson) varieties"

Definition

A holomorphic quasi-Hamiltonian G-space is a complex G-manifold M with a G-invariant two form ω and a G-equivariant map $\mu: M \rightarrow \mathfrak{g}$ (G acts on \mathfrak{g} by conjugation)

such that

$$\textcircled{1} \quad d\omega = \mu^*(\eta)$$

$$\textcircled{2} \quad \forall X \in \mathfrak{g} \quad \omega(\nu_X, \cdot) = \frac{1}{2} \mu^*(\Theta + \bar{\Theta}, X)$$

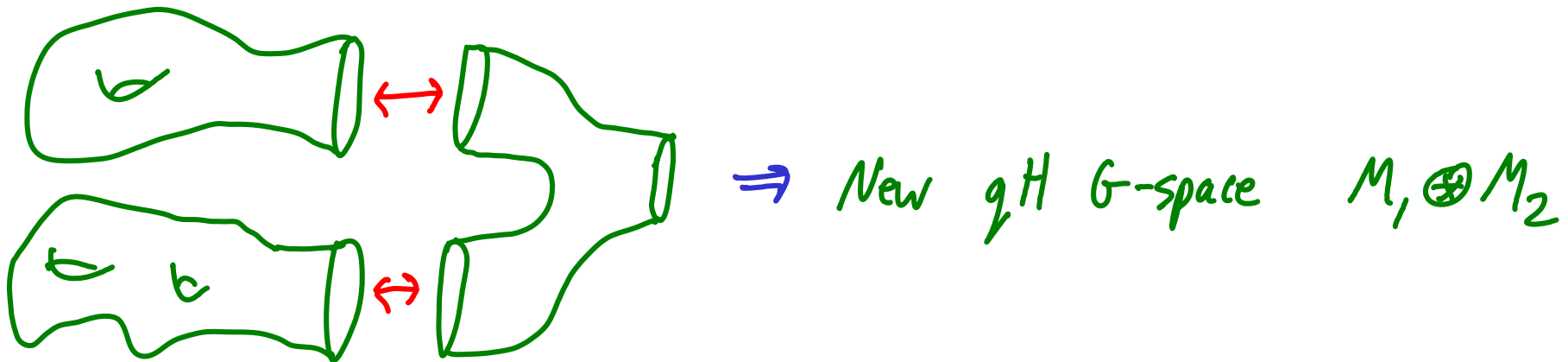
$$\textcircled{3} \quad \forall m \in M \quad \ker \omega_m \cap \ker d\mu = \{0\} \subset T_m M$$

where $\eta =$ biinvariant 3-form on G , $\Theta, \bar{\Theta}$ Maurer-Cartan forms on G

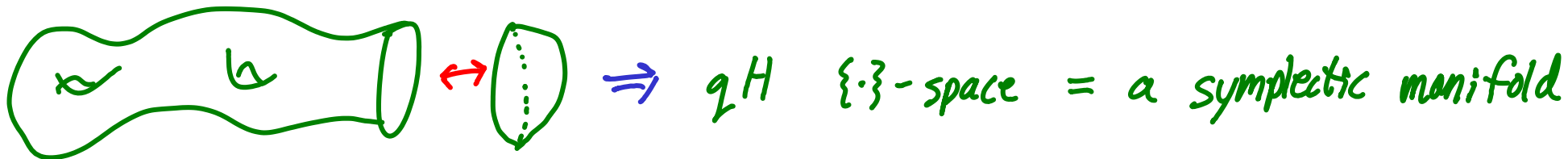
- These axioms are 'what we get from ω -d viewpoint'
- Multiplicative analogue of Hamiltonian G-space (with \mathfrak{g}^* -valued moment map)

Operations

① Can 'fuse' 2 qHamiltonian G-spaces:

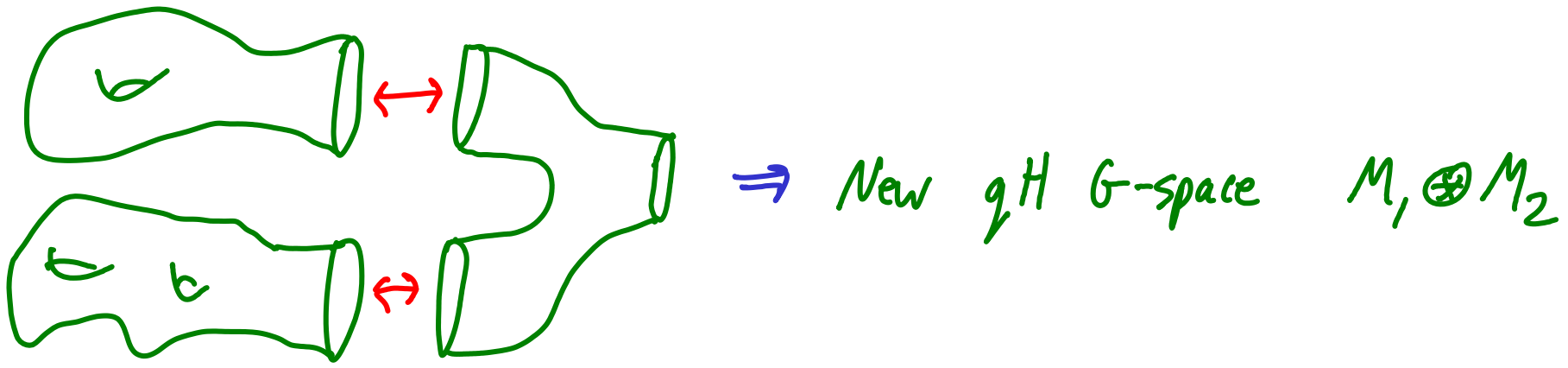


② & reduce:

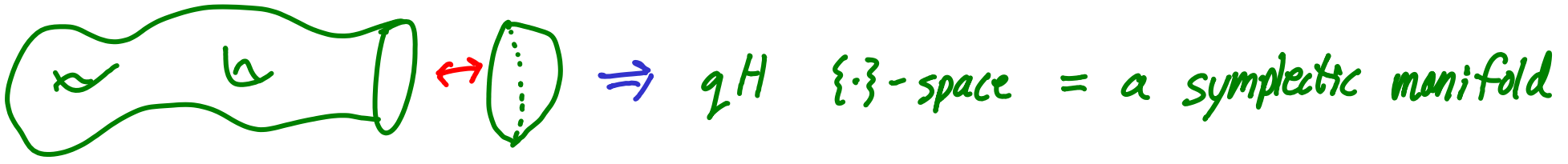


Operations

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② & reduce:



Basic examples

① Conjugacy classes $\mathcal{C} \subset G$

② $D = G \times G$ qH $G \times G$ space (double)

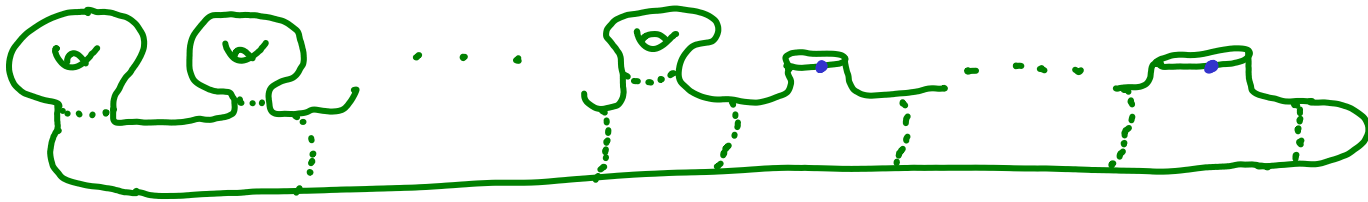


③ $1D = G \times G$ qH G-space (internally fused double)



Can construct all moduli spaces of holomorphic connections
on Riemann surfaces from these pieces:

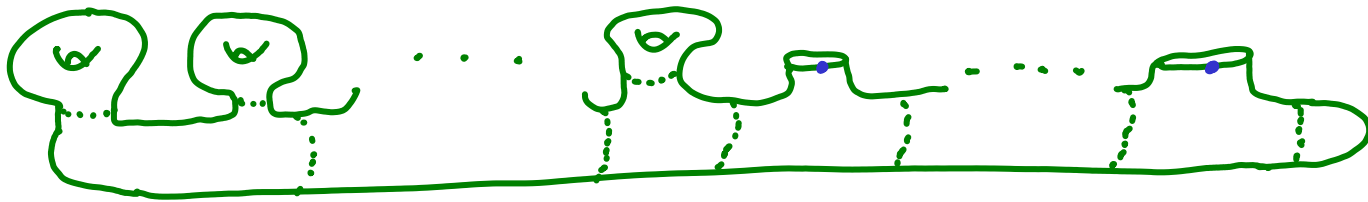
$$\underbrace{\mathbb{D} \otimes \dots \otimes \mathbb{D}}_g \otimes \underbrace{\mathbb{D} \otimes \dots \otimes \mathbb{D}}_m // G \cong \text{Hom}(\pi, G)$$



$$\mu^{-1}(e) / G^m \cong \left\{ (A, B, M) \mid \prod_{i=1}^g [A_i, B_i] \prod_{i=1}^m M_i = 1, M_i \in \mathcal{C}_i \right\} / G$$

Can construct all moduli spaces of holomorphic connections on Riemann surfaces from these pieces:

$$\underbrace{\mathbb{D} \otimes \dots \otimes \mathbb{D}}_g \otimes \underbrace{\mathbb{D} \otimes \dots \otimes \mathbb{D}}_m // G \cong \text{Hom}(\pi, G)$$



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Aim: New pieces to construct irregular Betti spaces?

(have "irreg. Atiyah-Bott" from 1999)

Fission spaces

Choose $P_{\pm} \subset G$ opposite parabolics

$H = P_+ \cap P_-$ Levi subgroup

$U_{\pm} \subset P_{\pm}$ unipotent radicals

Thm (- '02, '09, '11)

The "fission space" $G A_H^r := G \times (U_+ \times U_-)^r \times H$

is a quasi-hamiltonian $G \times H$ space

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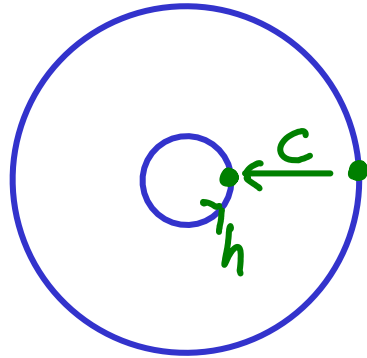
is a quasi-hamiltonian $G \times H$ space

- moment map $\mu(C, s_1, \dots, s_{2r}, h) = (C^{-1} h s_{2r} \cdots s_1 C, h^{-1})$
- $(U_+ \times U_-)^r \cong$ Stokes data of connections with $Q = \frac{A}{z^r}$, $C_G(A) = H$

Picture

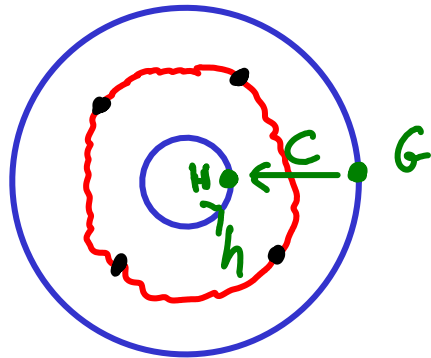
If $P_{\pm} = G = H$

$G \mathcal{A}_H = G \times G$ is the double
 \downarrow
 (C, h)



$$\mu = (C^{-1}hC, h^{-1})$$

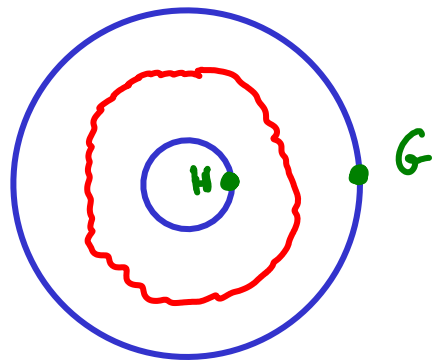
General case can be pictured similarly (breaking group from G to H)



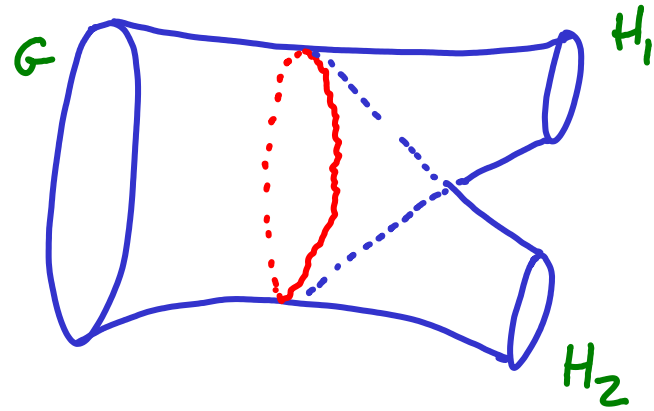
$$\mu = (C^{-1}hS_{2r} \dots S_1, C, h^{-1})$$

Typically H is a product eg. $H = H_1 \times H_2$

- can glue on both a qH H_1 -space & a qH H_2 -space



\cong



"fission" operation (\neq fusion)

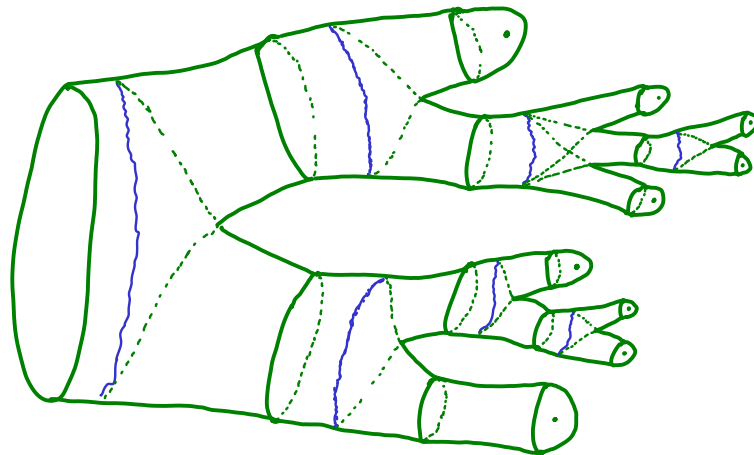
$$\text{If } Q = \frac{A_r}{z^r} + \dots + \frac{A_1}{z}$$

Define $G = H_r \supset H_{r-1} \supset \dots \supset H_0 = H \supset T$

$$\text{via } H_{i-1} = C_{H_i}(A_i)$$

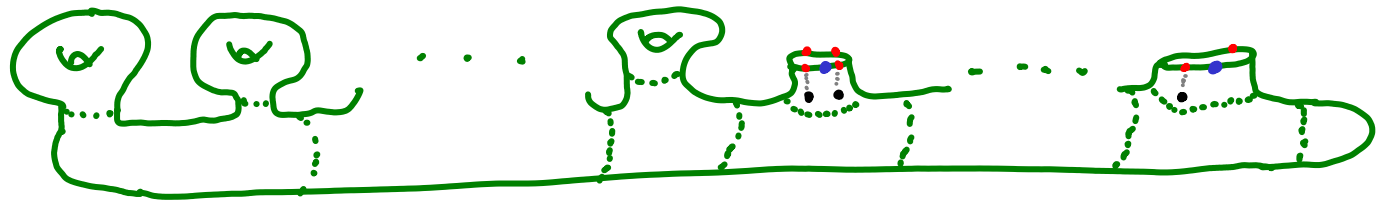
Then $A(Q) := G \times \{\text{Stokes data for } Q\} \times H$ obtained by gluing

$$A(Q) \cong G \overset{r}{A}_{H_{r-1}} \rightsquigarrow_{H_{r-1}} A_{H_{r-2}} \rightsquigarrow \dots \rightsquigarrow_{H_1} A_H$$



If Σ an irregular curve :

$$\text{Hom}_{\mathcal{S}}(\pi, \mathcal{G}) \cong \underbrace{\mathbb{D} \otimes \cdots \otimes \mathbb{D}}_g \otimes A(Q_1) \otimes \cdots \otimes A(Q_m) // \mathcal{G}$$



$$\mu^{-1}(e) / \underline{H} \cong \left\{ (A, B, C, h, S) \mid \prod_{i=1}^g [A_i, B_i] \prod_{i=1}^m \mu_i = 1, h_i \in \mathcal{C}_i \right\} / \underline{H}$$

$$\mu_i = C_i^{-1} h_i \cdots S_2^{(i)} S_1^{(i)} C_i$$

But there are many other examples of fission varieties

- e.g. can glue surfaces Σ along their boundaries
(provided the groups H_i match up)
- can obtain all the so-called multiplicative quiver varieties

