Irregular connections, Dynkin diagrams & fission

P. Boalch (ENS Paris)

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Starting point [Hitchin's Frobenius manifold question (1995)]

- Dubrovin (1994) classified semisimple Frobenius manifolds in terms of the "Stokes data" of an operator of form

$$\frac{d}{dz} - \left(\frac{U}{z^2} + \frac{V}{z} \right)$$

U, V $n \times n$ matrices, U diagonal with distinct eigenvalues
V skew-symmetric

e.g. $QH^*(\mathbb{CP}^2)$ is a 3d ss Frobenius manifold with Stokes matrix

$$\begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

- <u>Qn</u> Can we obtain a sharp classification in terms of monodromy data of logarithmic connections on $\mathbb{P}^1$?

- Ans Yes — have isomorphism between moduli spaces of such irreg connections & various moduli spaces of log. connections

  (via twisted Fourier–Laplace transform)

  choose good twist to see how to characterise log. con$^{\text{n}}$s

- Such isoms are braid group equivariant
- & relate natural complex symplectic / Poisson structures

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  (Dubrovin 1994)
  choose good twist to see how to characterise log. con^{ns}

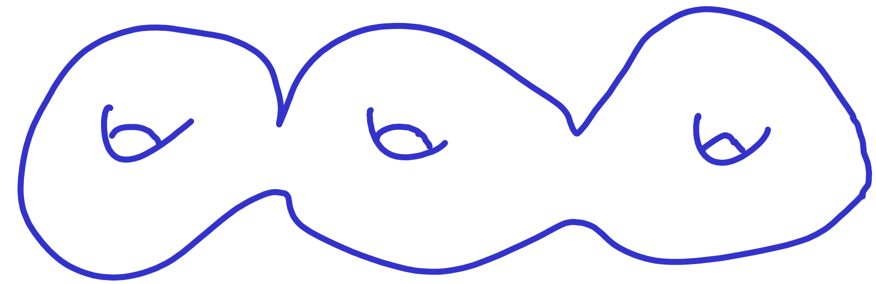
- Such isoms are braid group equivariant
- & relate natural complex symplectic / Poisson structures

My (contrary) question: Rather than try to convert questions about irregular connections into the well understood logarithmic / regular singular world, why not try to extend existing technology to the irreg. case?
— Clearly "most" connections on curves are irregular...

Themes:

- nonlinear symplectic braid group actions
- (wild) non-abelian Hodge corresp. on curves & hyperkähler moduli spaces
- understanding such moduli spaces on $\mathbb{P}^1$ $\left(\mathcal{M}^*_{DR} \subset \mathcal{M}_{DR}\right)$
- symplectic & quasi-Hamiltonian structures on Betti data

smooth projective curve X



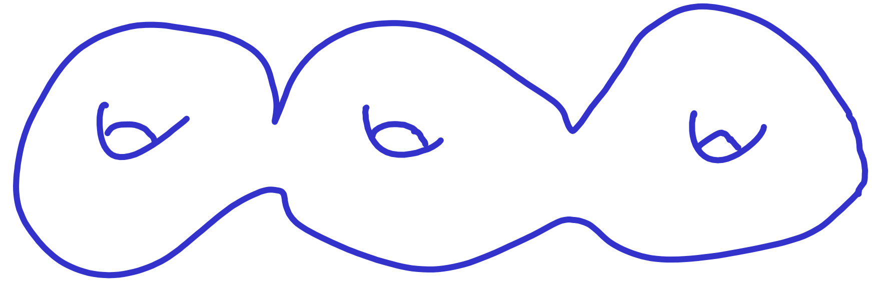
$\mathcal{M} = H^1(X, G)$ $\quad G$ connected complex reductive gp

3 viewpoints :

- $\mathrm{Hom}(\pi_1(X), G)/G$
- G-bundles with holomorphic connections
- G-bundles with Higgs fields

$\Phi \in H^0(X, \mathrm{ad}P \otimes K)$

smooth projective curve X



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3 viewpoints :

- $\mathrm{Hom}(\pi_1(X), G)/G$ $\quad \mathcal{M}_{Betti}$
- G-bundles with holomorphic connections $\quad \mathcal{M}_{DR}$
- G-bundles with Higgs fields $\quad \mathcal{M}_{Dol}$

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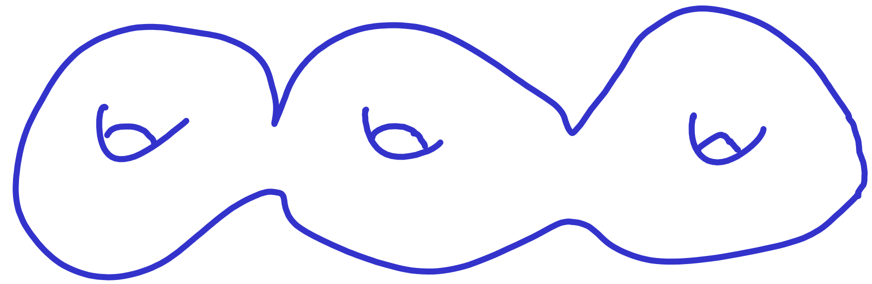
[Add stability conditions (or framings) to get nice moduli spaces]

smooth projective curve X



$$\mathcal{M} = H^1(X, G)$$

G connected complex reductive gp

- $\mathrm{Hom}(\pi_1(X), G)/G$ $\qquad \mathcal{M}_{Betti}$

Riemann-Hilbert $\Updownarrow$

- G-bundles with holomorphic connections $\qquad \mathcal{M}_{DR}$

Nonabelian Hodge $\Updownarrow$

- G-bundles with Higgs fields $\qquad \mathcal{M}_{Dol}$

$\Phi \in H^0(X, \mathrm{ad}P \otimes K)$

[Add stability conditions (or framings) to get nice moduli spaces]

E.g. $G = \mathbb{C}^*$

$$\mathcal{M}_{Dol} = T^*\mathrm{Jac}(X), \quad \mathcal{M}_{Betti} \cong (\mathbb{C}^*)^{2g}, \quad \mathcal{M}_{DR} \longrightarrow \mathrm{Jac}(X)$$

Extend to meromorphic connections ($G = GL_n$)

1 Regular singularities ~ open Riemann surfaces

Nonabelian Hodge corresp. (Simpson 1990)

Hyperkahler structures (Nakajima 1996)

Rough picture

- $\mathcal{M}_{Betti} = \mathrm{Hom}(\pi_1(X), G)/G$

fix conjugacy class of monodromy around each puncture to fix symplectic leaves

- $\mathcal{M}_{DR}$ = G-bundles + log connections

- $\mathcal{M}_{Dol}$ = G-bundles + Higgs fields with simple poles

fix adjoint orbits of residues to fix leaves

② Irregular singularities

Hyperkahler structures & non-abelian Hodge corresp. (Biquard-B.)* 2004

(holom. symplectic structures PB '99) (one direction by Sabbah 1999)

Rough picture:

- $\mathcal{M}_{DR}$: fix formal type $(G[\![z]\!])$ to fix symplectic leaves
- $\mathcal{M}_{Betti}$: include Stokes data too
- $\mathcal{M}_{Dol}$: mero. Higgs bundles (ACIHS by Bottacin, Markman 1994)

* only unramified formal types considered – general case is similar (cf. Witten 2007)

Braiding/Mapping class group actions

"Isomonodromy = Nonabelian Gauss-Manin connection"

(extended to irregular case)

Braiding/Mapping class group actions

"Isomonodromy = Nonabelian Gauss-Manin connection"

(PB 2001)

(extended to irregular case)

1900's (higher) Painlevé equations

~ 1980 Sato Miwa Jimbo Ueno ...

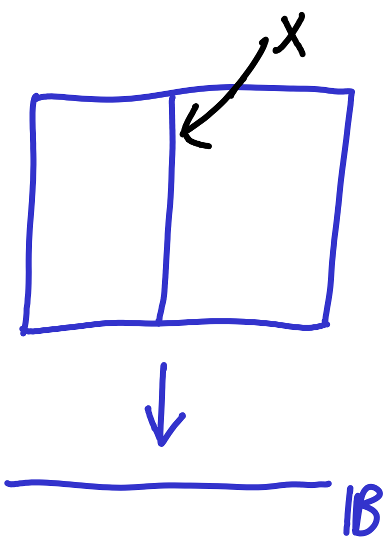
Simpson 1994

Braiding/Mapping class group actions

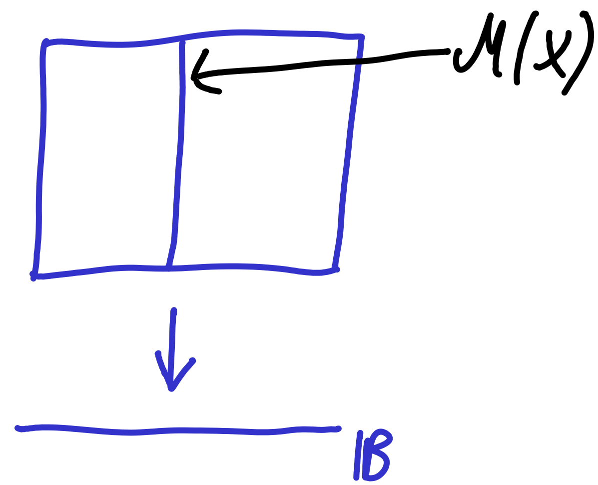
"Isomonodromy = Nonabelian Gauss-Manin connection"

(extended to irregular case)

Reg. case:



$\Longrightarrow$



family of curves with marked points

'family' of moduli spaces

- nonlinear fibre bundle with flat algebraic connection
- Betti spaces form a local system of varieties

What is the base $\mathbb{B}$ in the irregular case?

Def^n Fix $T \subset G$

An "irregular curve" is

- A smooth complex curve Σ, with
- distinct marked points $a_1, \dots, a_m$, and
- an irregular type Q at each marked point

$$Q = \frac{A_r}{z^r} + \cdots + \frac{A_1}{z} \in \mathfrak{t}((z)) / \mathfrak{t}[\![z]\!]$$

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Look at deformations such that:

- Σ remains smooth, • points a_i remain distinct
- Pole Order $(\alpha \circ Q)$ does not change $(\forall \text{ roots } \alpha \in R \subset \mathfrak{t}^*)$

e.g. if $A_r \in \mathfrak{t}_{reg}$

Basic fact:

If we consider meromorphic connections A on G-bundles $/\Sigma$ with poles at $\{a_i\}$ s.t. locally:

$$A \underset{G[\![z]\!]}{\sim} dQ + \Lambda \frac{dz}{z} \quad \text{at each pole}$$

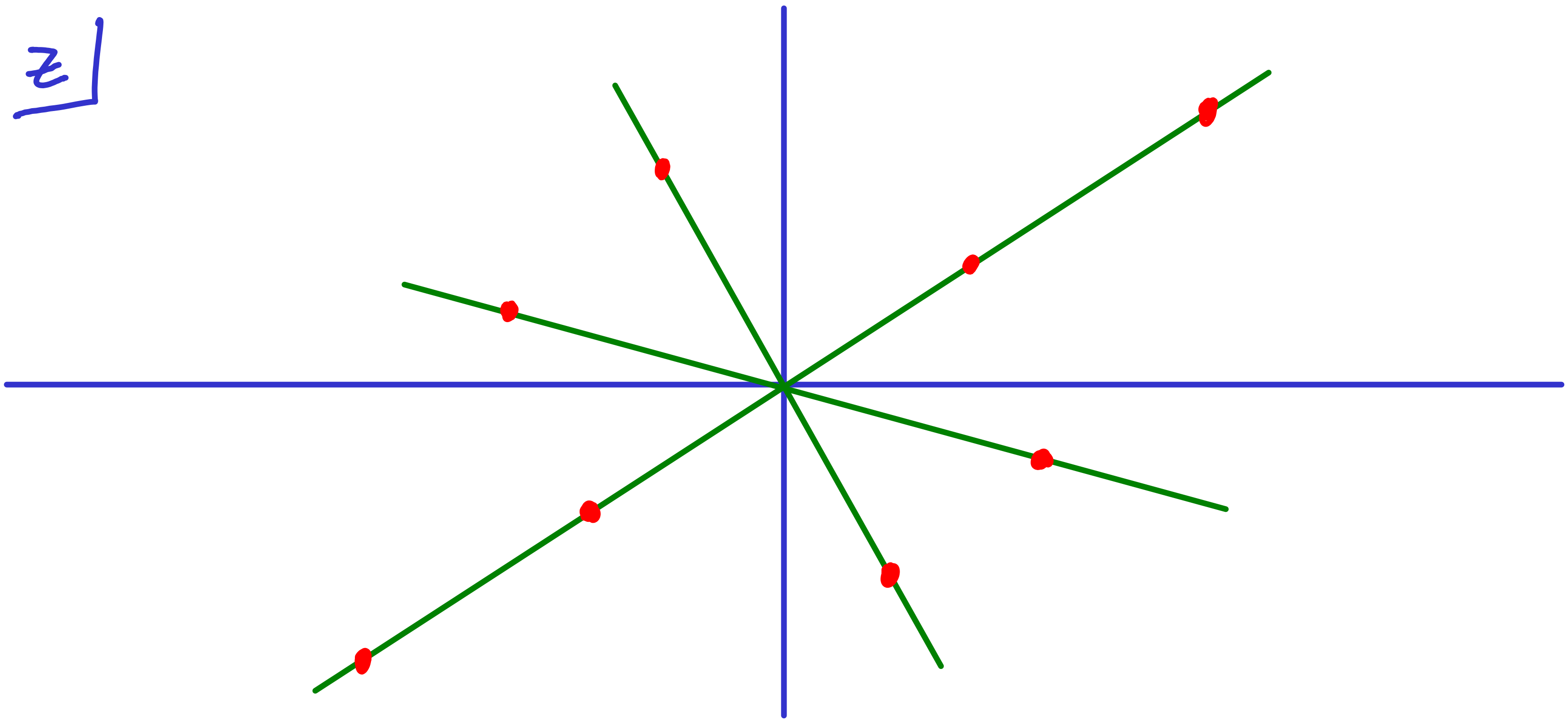
Λ: holom.

Then again get flat algebraic con^n on family of moduli spaces over the space of (irregular) curve deformations

(cf. Jimbo-Miwa-Ueno '81 (GL_n), PB '02 (other G))

Simplest example (PB '02) $\quad r=1, \quad Q = \frac{-A_1}{z}, \quad A_1 \in t_{reg}$

Plot roots on z-plane: $\langle A_1, R \rangle \subset \mathbb{C}^*$



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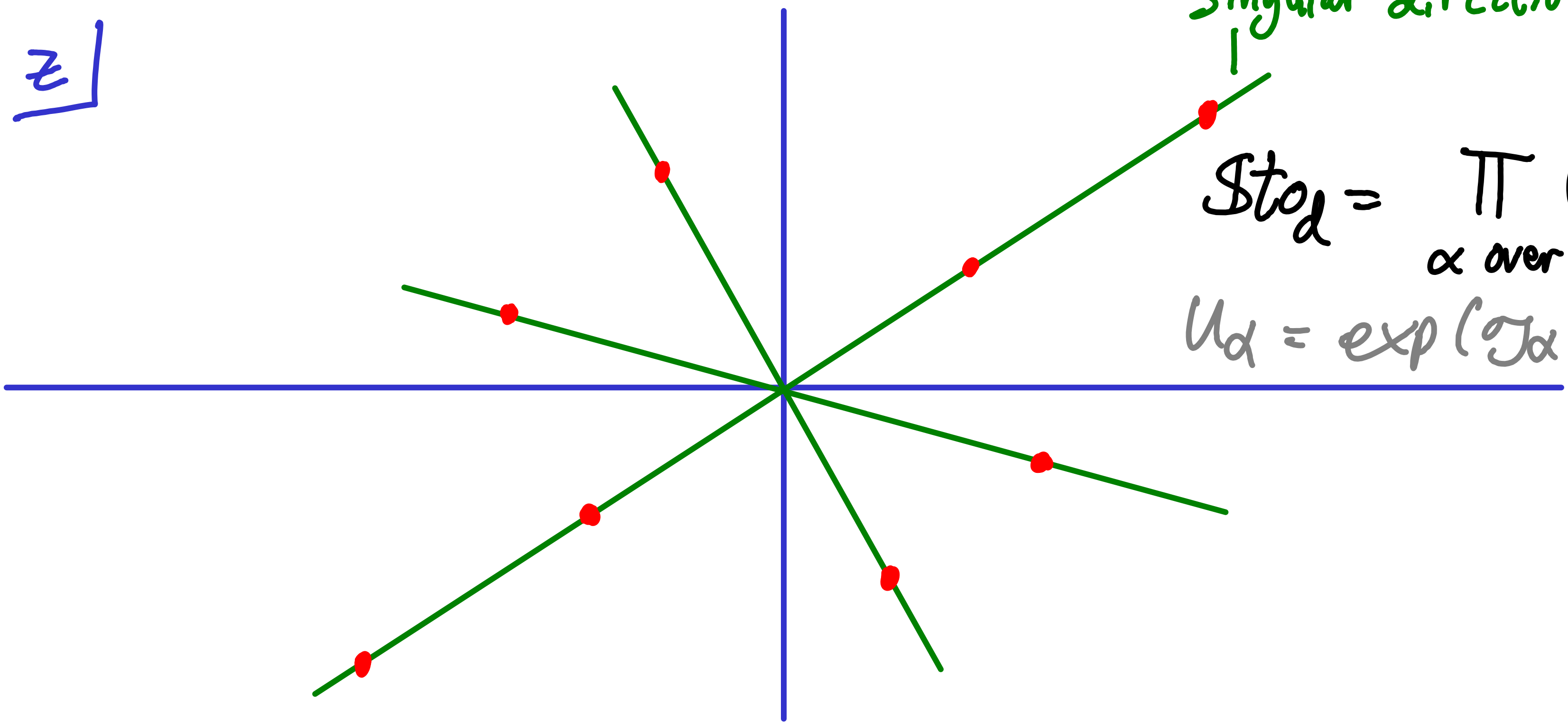

$$\mathbb{S}to_d = \prod_{\alpha \text{ over } d} U_\alpha \subset G$$

$U_\alpha = \exp(\mathfrak{g}_\alpha)$ root gp

$$\{\text{Stokes data}\} = \prod_d \mathbb{S}to_d$$

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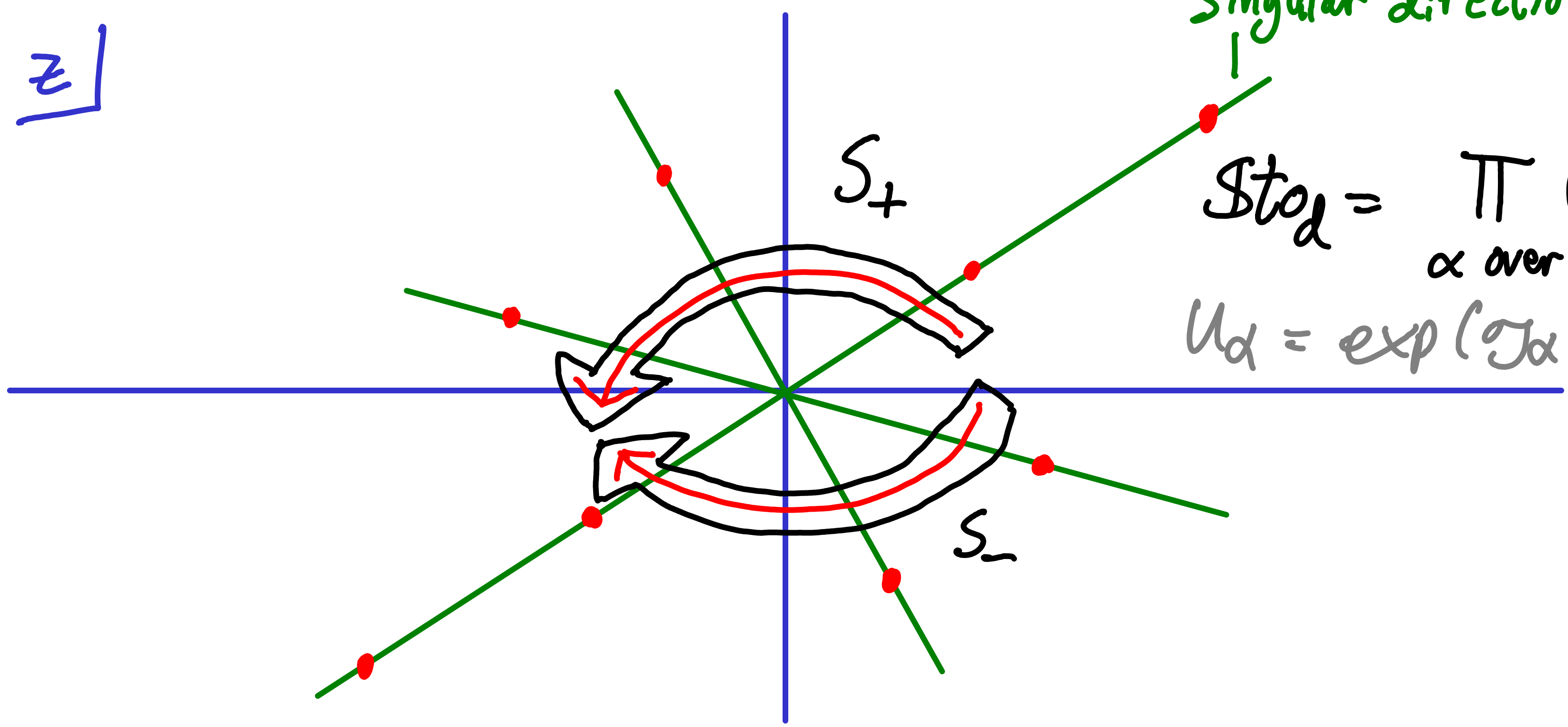
singular directions d



$$\mathrm{Sto}_d = \prod_{\alpha \text{ over } d} U_\alpha \subset G$$

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$$\{\text{Stokes data}\} = \prod_d \mathrm{Sto}_d \cong \underbrace{U_+ \times U_-}_{\text{unipotent radicals of opposite Borels}} \ni (S_+, S_-)$$

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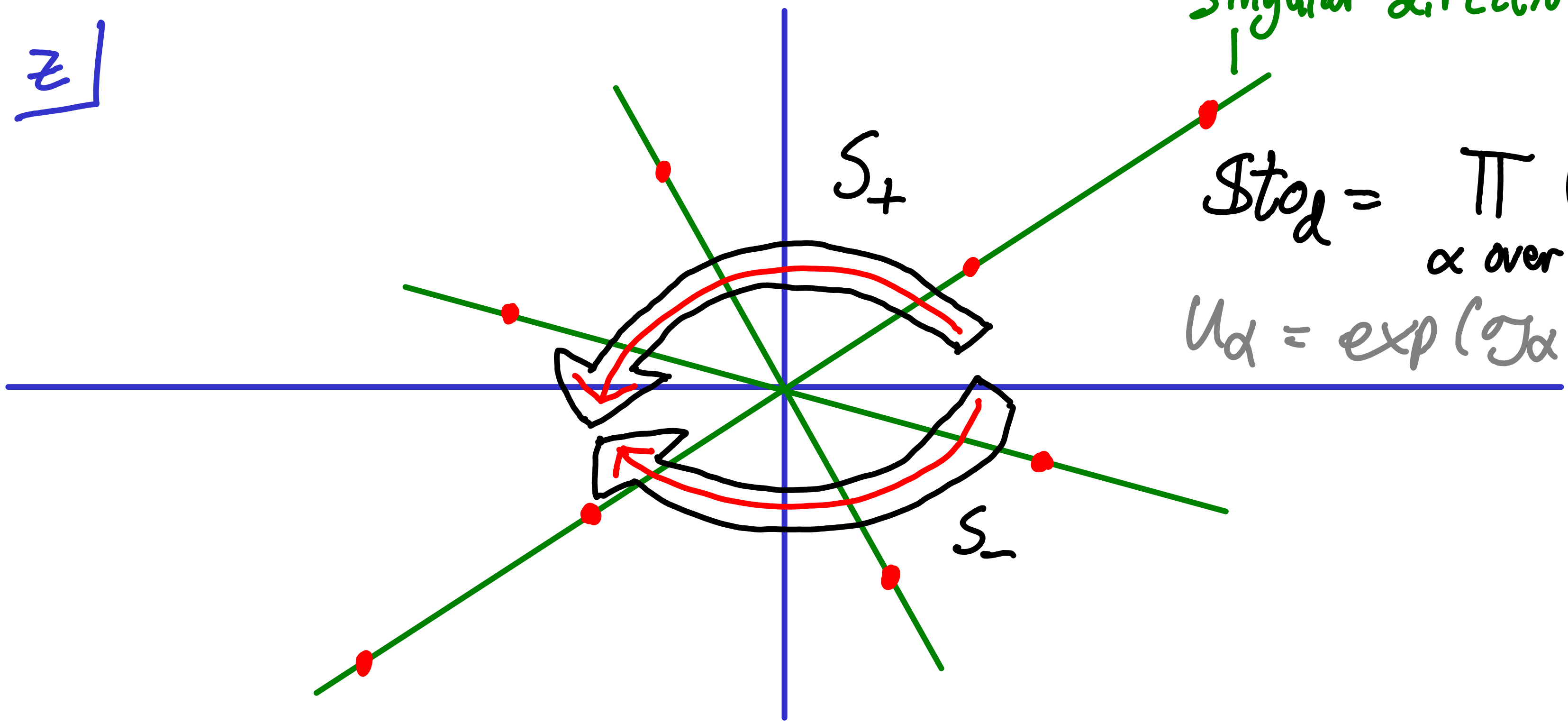

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Isomonodromy: Vary $A_1 \in \mathfrak{t}_{reg}$ & keep $S_\pm$ const. (locally)

In this example the resulting braid gp action had been previously seen:

$$
\begin{tikzcd}
& U_q\mathfrak{g} & \\
\mathrm{Fun}(G^*) \arrow[ur] & & U\mathfrak{g} \arrow[ul] \\
& \mathrm{Fun}(\mathfrak{g}^*) = \mathrm{Sym}(\mathfrak{g}) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

$G^* \cong T \ltimes (U_+ \times U_-)$ dual Poisson-Lie group (Drinfeld)

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$B_{\mathfrak{g}} \circlearrowleft U_q\mathfrak{g}$ Quantum Weyl group action (Lusztig, Soibelman ...)

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\mathcal{B}_{\mathfrak{g}} \circlearrowright & \mathrm{Fun}(G^*) \arrow[u] & & U\mathfrak{g} \arrow[ull] \\
& & \mathrm{Fun}(\mathfrak{g}^*) = \mathrm{Sym}(\mathfrak{g}) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

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Thm (- '02)

The DKP action arises from isomonodromy ($U_+ \times U_-$ = Stokes data)

- Purely geom. origin (not just explicit generators)
- $U_q\mathfrak{g}$ quantizes a moduli space of mero. connections

General case is similar; space of Stokes data more complicated:

$$\{\text{Stokes data}\} = \prod_{d \in \mathbb{A}} \mathbb{S}\mathrm{to}_d$$

General case is similar; space of Stokes data more complicated:

$$\{\text{Stokes data}\} = \prod_{d \in \mathbb{A}} \mathbb{S}to_d$$

- singular directions $\mathbb{A} = \left\{ d \in S^1 \;\middle|\; \begin{array}{l} e^{\alpha \circ Q(z)} \text{ has max. decay} \\ \text{as } z \to 0 \text{ along } d \text{ for some root } \alpha \end{array} \right\}$ $\circledast$

- $\mathbb{S}to_d = \prod_{\alpha \in R(d)} \exp(\mathfrak{g}_\alpha) < G$ (unipotent subgroup)

- $R(d) = \{\text{roots s.t } \circledast \text{ holds in direction } d\} < R$

Guide to moduli spaces on $\mathbb{P}^1$

Typically $\mathcal{M}^* \subset \mathcal{M}$

open part where bundle holom. trivial / $\mathbb{P}^1$

& $\mathcal{M}^*$ again a complete hyperkahler manifold

"approximation" to more transcendental metric on $\mathcal{M}$

Classical hyperkahler mfds

1. Complex coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$

(Kronheimer, Biquard, Kovalev)

If pole divisor $2(0) + (\infty) \subset \mathbb{P}^1$

have examples where $\mathcal{M}^* \cong \mathcal{O} /\!\!/\!\!/ T_K$

[$\mathcal{M}_{Betti} = \mathcal{L} /\!\!/ T$, $\mathcal{L} \subset G^*$ symplectic leaf]

($T_K \subset T$ compact torus)

② T^*G (Kronheimer)

If pole divisor $2(0) + 2(\infty) \subset \mathbb{P}^1$

have examples where $\mathcal{M}^* \cong T_K \backslash\!\backslash\!\backslash T^*G /\!/\!/ T_K$

$$\mathcal{M}_{Betti} = T \backslash\!\backslash \Gamma /\!/ T$$

$\Gamma \subset (G \times G^*)^2$ Lu-Weinstein double sympl. groupoid

③ ALE spaces   deformations of $\widetilde{\mathbb{C}^2/\Gamma}$

(Eguchi-Hanson, Gibbons-Hawking, Hitchin, Kronheimer)

$\dim_{\mathbb{R}} = 4$ (gravitational instantons / quaternionic curves)

$\Gamma \subset SU_2$ finite $\longleftrightarrow$ ADE affine Dynkin graph

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Fact $\quad$ In cases $E_8, E_7, E_6, D_4, A_3, A_2, A_1$

have $\mathcal{M}$ s.t $\mathcal{M}^{\#} \subset \mathcal{M}$ is corresponding ALE space

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Fact In cases E_8, E_7, E_6, D_4 (logarithmic realisations), A_3, A_2, A_1 (only irregular realisations)

have $\mathcal{M}$ s.t $\mathcal{M}^* \subset \mathcal{M}$ is corresponding ALE space

	Pole orders
A_3	$2+1+1$
A_2	$3+1$
A_1	4

– Okamoto found in 1987 the corresponding affine Weyl groups are the sym gps of the corresponding Painlevé equations

Rough classification (of $\mathcal{U}$'s) in $\dim_{\mathbb{C}} = 2$:

E_8 $\quad$ E_7 $\quad$ E_6 $\quad$ D_4 $\quad$ $A_3 = D_3$ $\quad$ D_2 $\quad$ D_1 $\quad$ (D_0)

A_2 $\quad$ A_1 $\quad$ (A_0)

reg $\leftarrow$ | $\rightarrow$ irreg

4 (Nakajima) Quiver varieties

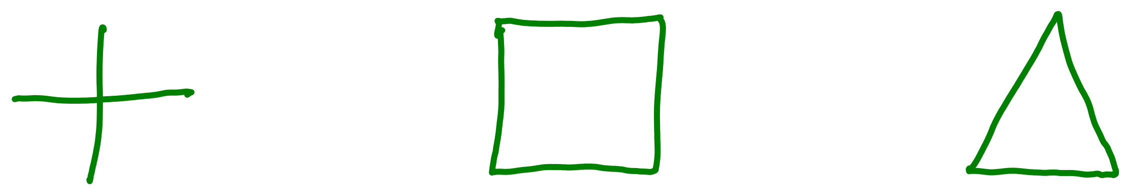
$$\overset{V}{\circ}\!\!-\!\!-\!\!-\!\!-\!\!\overset{W}{\circ}$$

$\mathrm{Hom}(V,W) \oplus \mathrm{Hom}(W,V)$ is hyperkahler $U(V) \times U(W)$ space

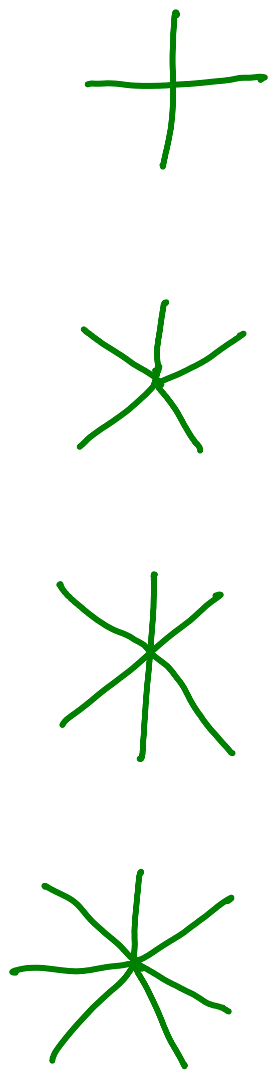
Graph = ADE dynkin graph $\Rightarrow$ ALE space (Kronheimer)
else in general get higher dim^n hyperkahler mfd (or empty)

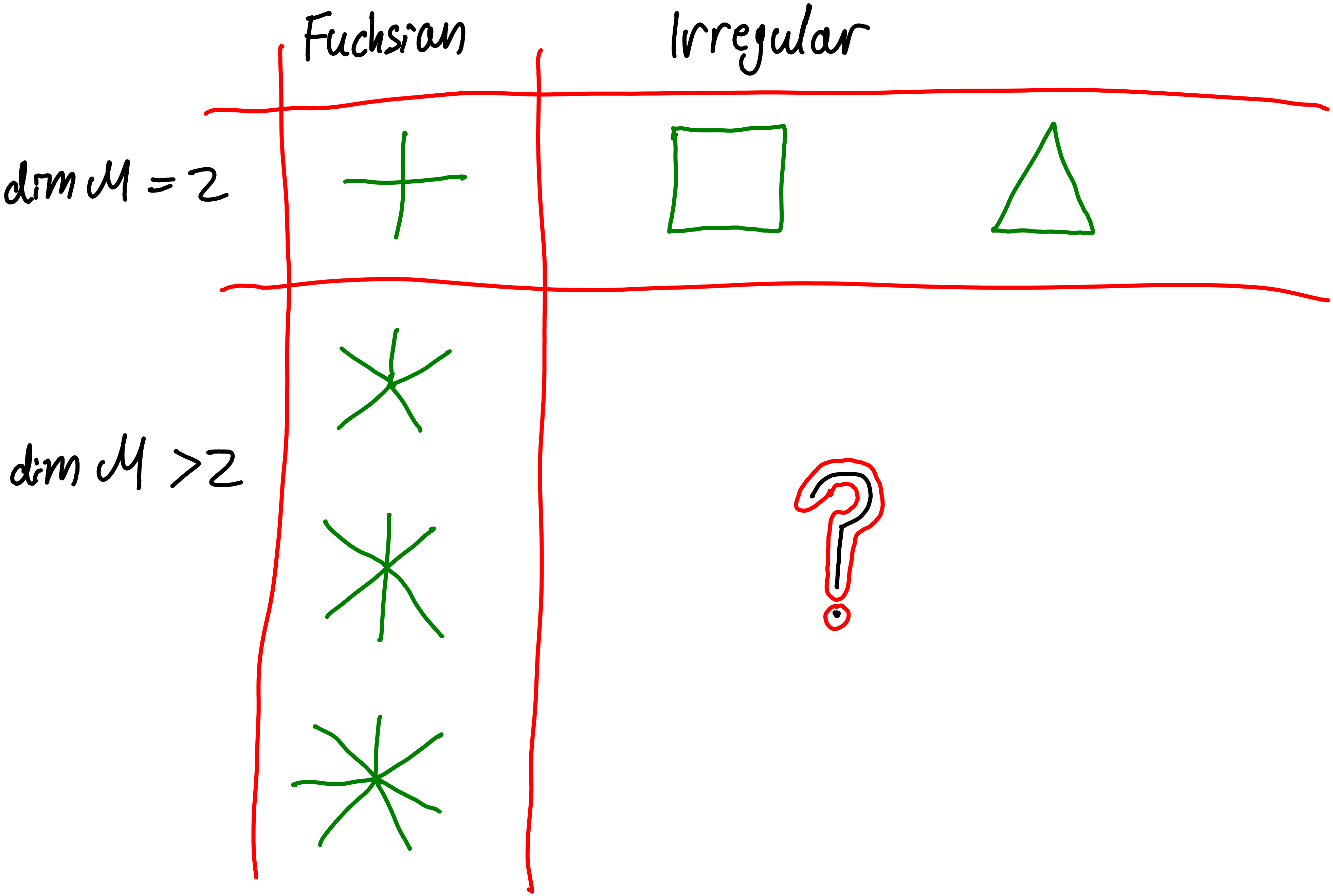
— lets consider simply-laced cases

Recall Okamoto showed the Painlevé equations 4, 5, 6 have affine Weyl group symmetries of type A_2, A_3, D_4 resp.



Recall Crawley-Boevey related moduli spaces of Fuchsian systems to star-shaped quivers (building on Kraft-Procesi, Nakajima, ...)

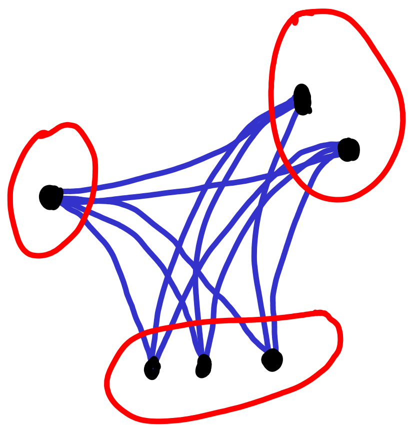



Fuchsian
Irregular
dim M = 2
dim M > 2
?

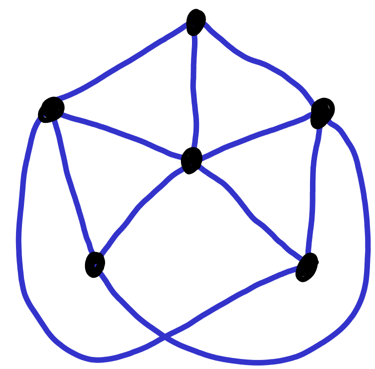
Thm (– '08)

Can take any complete k-partite graph (for any k)

E.g.



$\cong$



$\Gamma(3,2,1)$

– get action of corresponding (not necessarily affine) Kac-Moody Weyl group

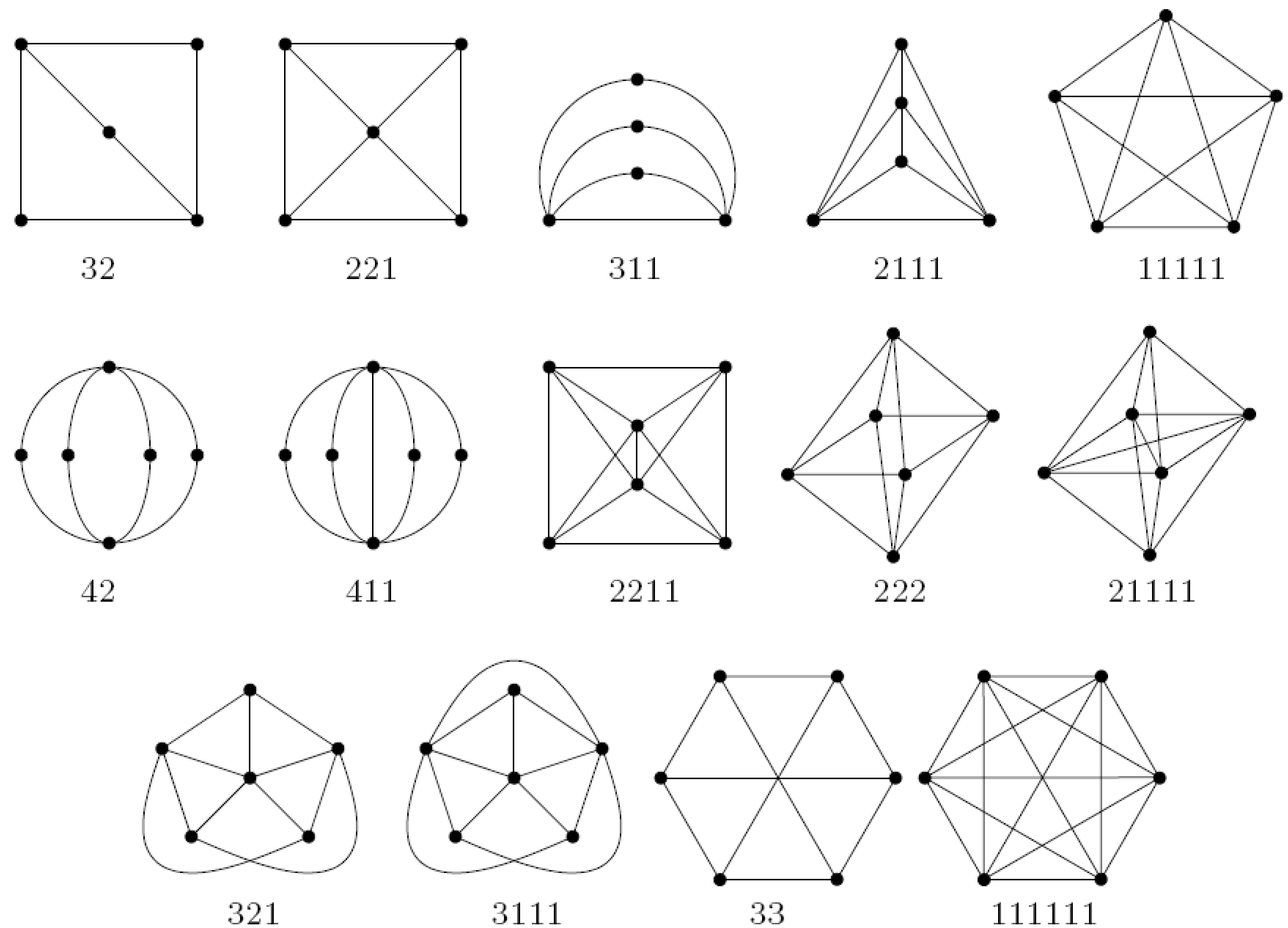


FIGURE 1. Graphs from partitions of $N \leq 6$
(omitting the stars $\Gamma(n,1)$ and the totally disconnected graphs $\Gamma(n)$)

Symplectic geometry of $\mathcal{M}_{Betti}$

Long story in non-singular case:

Atiyah-Bott (1983) Yang Mills equations over Riemann surfaces

Goldman (1984) Symplectic nature of π_1's of surfaces

⋮

Alekseev-Malkin-Meinrenken (1998) Lie group valued moment maps

X compact Riemann surface

G connected complex reductive group, Lie algebra $\mathfrak{g}$, $(,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$

$\mathcal{A} = \{ C^\infty \text{ connections on trivial } G \text{ bundle over } X \}$

$\mathcal{G} = C^\infty(X, G)$

$\mathcal{A}$ is symplectic & $\mathcal{G}$ acts on $\mathcal{A}$ with moment map

$$\mu(\alpha) = \text{curvature}(\alpha) + \alpha|_{\partial X}$$

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If no boundary components:

$$\mathcal{A} /\!/ \mathcal{G} = \mu^{-1}(0)/\mathcal{G} = \mathcal{A}_{flat}/\mathcal{G} \cong \mathcal{M}_{Betti}$$

If one boundary component:

$$1 \longrightarrow \mathcal{G}_\partial \longrightarrow \mathcal{G} \longrightarrow LG \longrightarrow 1$$

$g \in \mathcal{G}$ s.t $g|_{\partial X} = 1$

$C^\infty(\partial X, G)$

Curvature is moment map for $\mathcal{G}_\partial$

If one boundary component:

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$\mathcal{N} := \mathcal{A}/\!/\mathcal{G}_\partial = \mathcal{A}_{flat}/\mathcal{G}_\partial$ is still symplectic —

infinite dimensional Hamiltonian LG space

If one boundary component:

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$g \in \mathcal{G}$ s.t. $g|_{\partial X} = 1$ $\qquad$ $C^\infty(\partial X, G)$

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infinite dimensional Hamiltonian LG space

AMM Can encode in finite dimensional space:

Hamiltonian LG spaces $\longrightarrow$ quasi-Hamiltonian G-spaces

$\mathcal{N} \longmapsto \mathcal{N}/\Omega G$

– equivalence for compact G (w. conditions) $\qquad$ based loop group

Definition A holomorphic quasi-Hamiltonian G-space is a complex G-manifold M with a G-invariant two form ω and a G-equivariant map $\mu: M \longrightarrow G$ (G acts on G by conjugation) such that

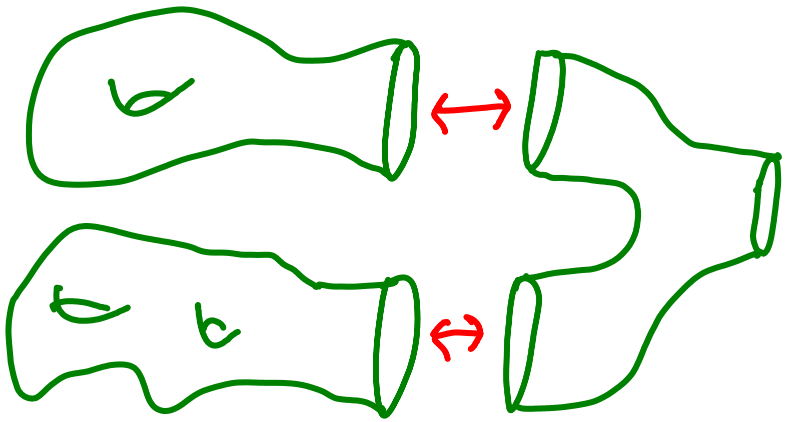
1. $d\omega = \mu^*(\eta)$
2. $\forall X \in \mathfrak{g} \quad \omega(v_X, \cdot) = \frac{1}{2}\mu^*(\theta + \bar{\theta}, X)$
3. $\forall m \in M \quad \ker \omega_m \cap \ker d\mu = \{0\} \subset T_m M$

where η = biinvariant 3-form on G, $\theta, \bar{\theta}$ Maurer-Cartan forms on G

- These axioms are 'what we get from ∞-d viewpoint'
- Multiplicative analogue of Hamiltonian G-space (with $\mathfrak{g}^*$-valued moment map)

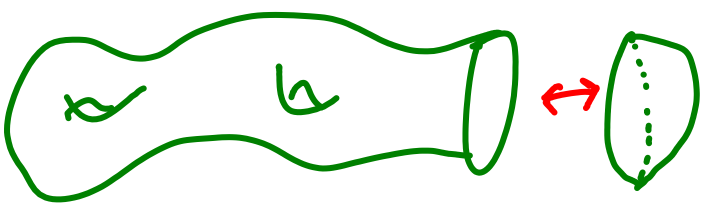
Operations

1. Can 'fuse' 2 q Hamiltonian G-spaces:



$\Rightarrow$ New qH G-space $M_1 \circledast M_2$

2. & reduce:



$\Rightarrow$ qH $\{\cdot\}$-space = a symplectic manifold

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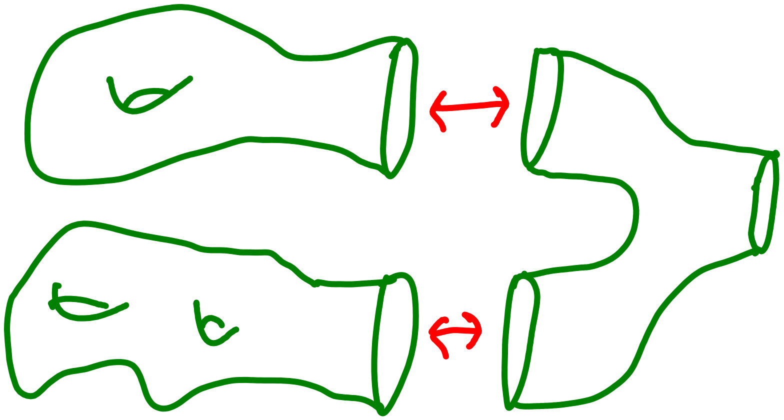

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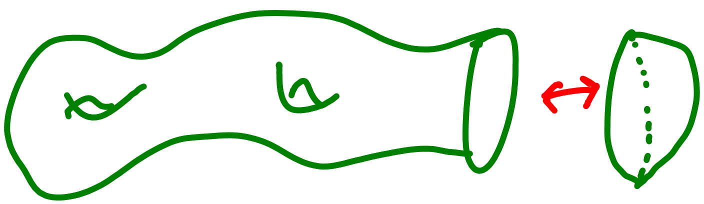
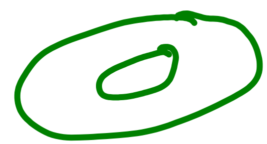


$\Rightarrow$ qH $\{\cdot\}$-space = a symplectic manifold

Basic examples

1. Conjugacy classes $\mathcal{C} \subset G$
2. $D = G \times G$ qH $G \times G$ space (double)

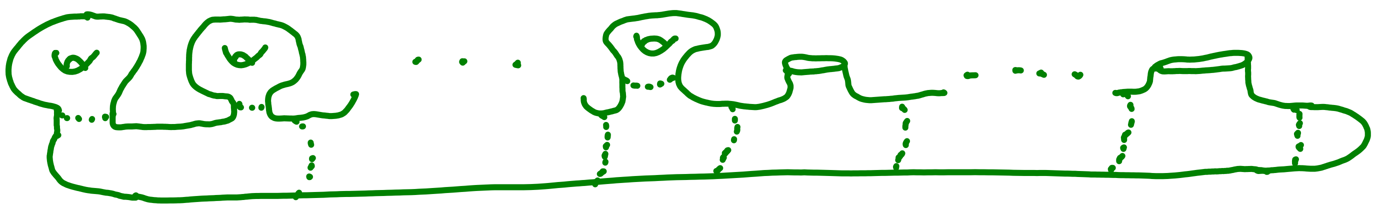


Annulus

3. $\mathbb{D} = G \times G$ qH G-space (internally fused double)

Can construct all moduli spaces of holomorphic connections on Riemann surfaces from these pieces:

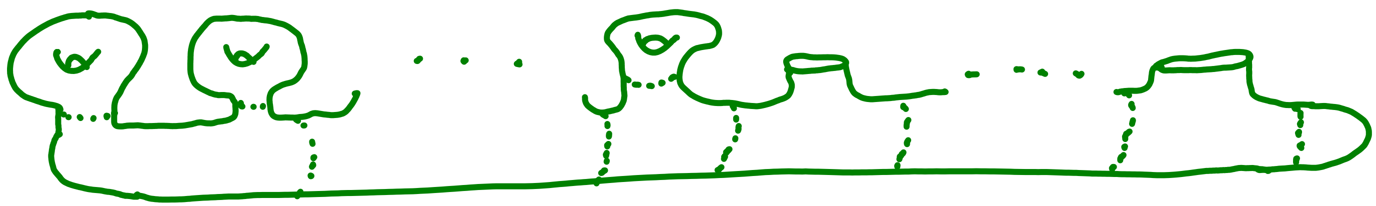
$$\underbrace{\mathbb{D} \circledast \cdots\cdots \circledast \mathbb{D}}_{g} \circledast \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!\!/ G$$



$$\left\{ (\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{M}) \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] \prod_{i=1}^{m} M_i = 1 \,,\; M_i \in \mathcal{C}_i \right\} / G$$

Can construct all moduli spaces of holomorphic connections on Riemann surfaces from these pieces:

$$\underbrace{\mathbb{D} \circledast \cdots \cdots \circledast \mathbb{D}}_{g} \circledast \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!/ G$$



$$\left\{ (\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{M}) \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] \prod_{i=1}^{m} M_i = 1 \,,\; M_i \in \mathcal{C}_i \right\} / G$$

Aim: New pieces to construct irregular Betti spaces?

(have "irreg. Atiyah-Bott" from 1999)

New pieces

Choose $P_\pm \subset G$ opposite parabolics

$H = P_+ \cap P_-$ Levi subgroup

$U_\pm \subset P_\pm$ unipotent radicals

Thm (– '02, '09, '11)

The space ${}_G\mathcal{A}^r_H = G \times (U_+ \times U_-)^r \times H$

is a quasi-Hamiltonian $G \times H$ space

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Thm (– '02, '09, '11)

The space $_G\mathcal{A}^r_H = G \times (U_+ \times U_-)^r \times H$

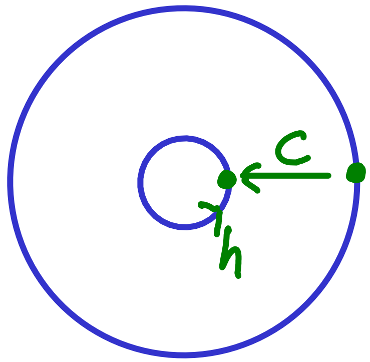
is a quasi-Hamiltonian $G \times H$ space

- moment map $\mu(C, S_1, \dots, S_{2r}, h) = (C^{-1} h S_{2r} \cdots S_1 C, h^{-1})$
- $(U_+ \times U_-)^r$ is Stokes data of connections with $Q = \frac{A}{z^r}$, $C_G(A) = H$
- all spaces of Stokes data mentioned earlier arise by gluing together such pieces

Picture

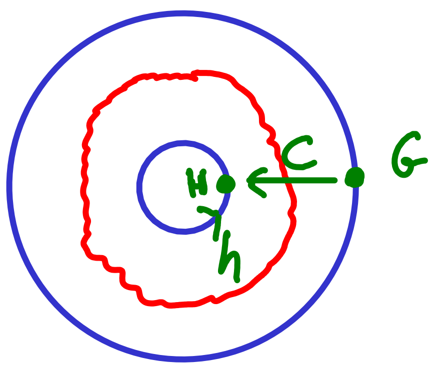
If $P_\pm = G = H$ $\quad {}_G\mathcal{A}_H = G \ltimes G$ is the double

$$\Downarrow$$
$$(c, h)$$



$$\mu = (c^{-1} h c,\ h^{-1})$$

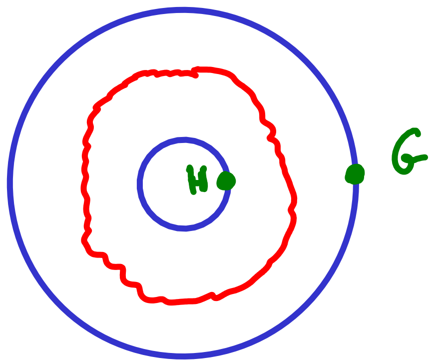
General case can be pictured similarly (breaking group from G to H)



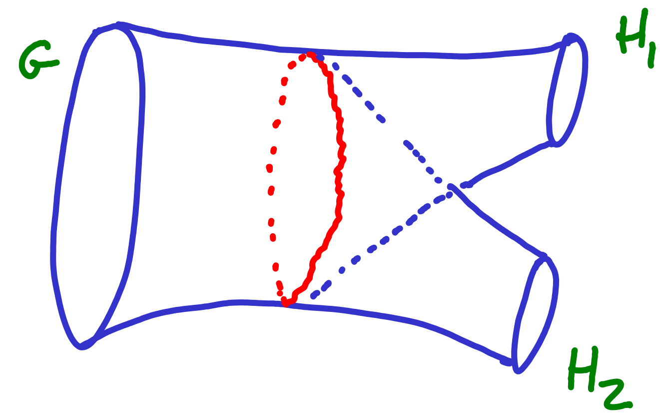
$$\mu = (c^{-1} h s_{2r} \cdots s_1 c,\ h^{-1})$$

Typically H is a product e.g. $H = H_1 \times H_2$

- can glue on both a qH H_1-space & a qH H_2-space



$\cong$



"fission" operation ($\neq$ fusion)

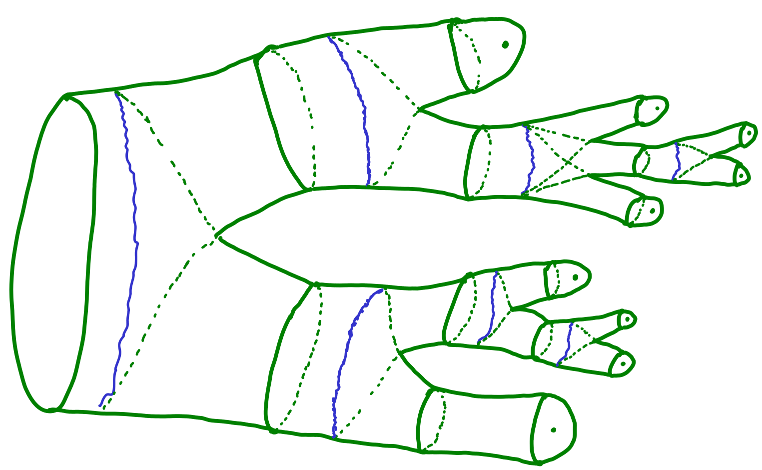
If $\quad Q = \dfrac{A_r}{z^r} + \cdots + \dfrac{A_1}{z}$

Define $\quad G = H_r \supset H_{r-1} \supset \cdots \supset H_0 = H \supset T$

via $\quad H_{i-1} = C_{H_i}(A_i)$

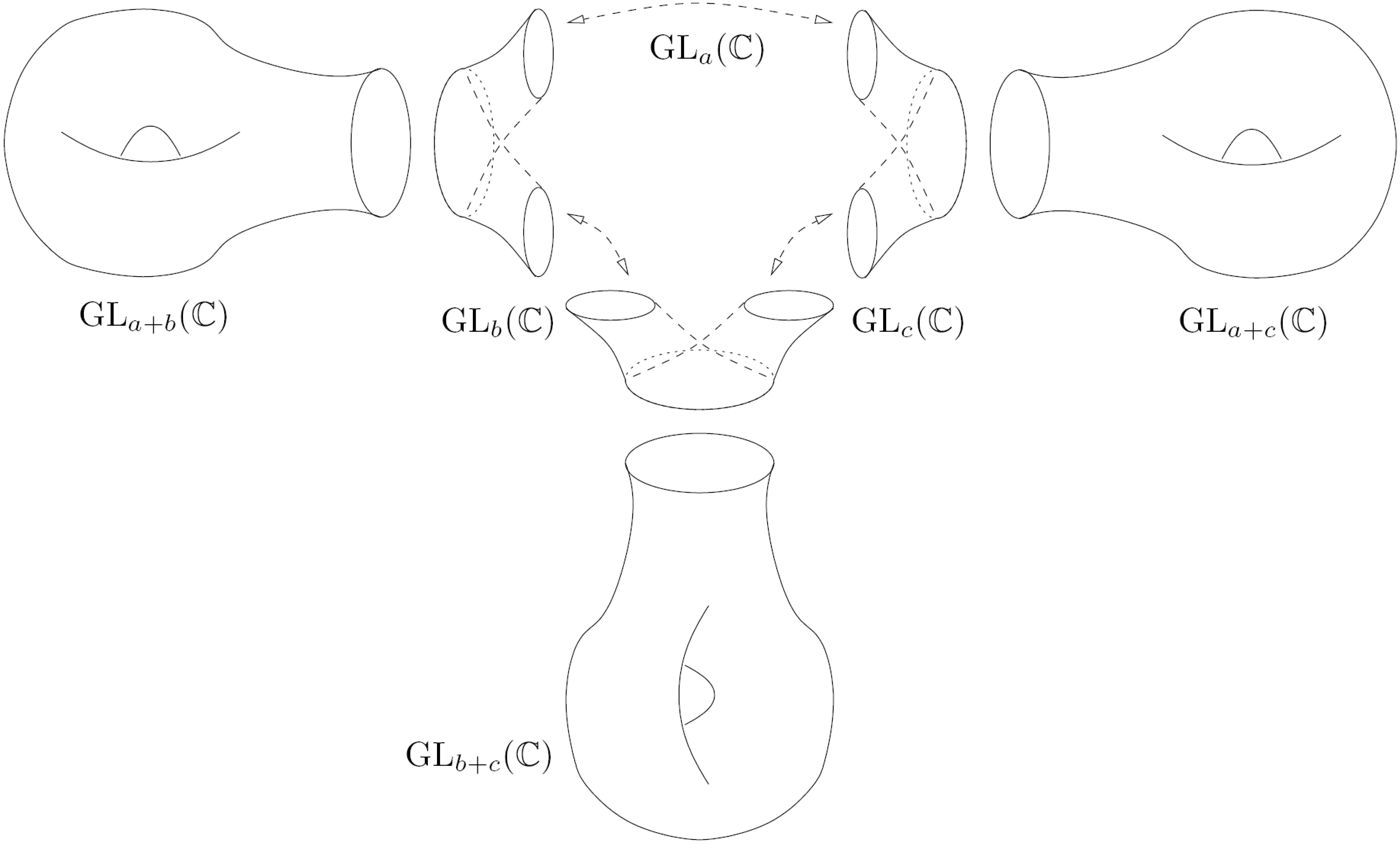
Then $\quad G \times \{\text{Stokes data for } Q\} \times H$ obtained by gluing

$$G\, \mathcal{A}^r_{H_{r-1}} \rtimes\!\!\!\!\subset {}_{H_{r-1}}\mathcal{A}_{H_{r-2}} \rtimes\!\!\!\!\subset \cdots \rtimes\!\!\!\!\subset {}_{H_1}\mathcal{A}_H$$



Finally observe the class of spaces generated by gluing together such pieces goes beyond moduli spaces of Betti data of mero. connections on curves

(& strictly includes all the "multiplicative quiver varieties")



$\mathrm{GL}_a(\mathbb{C})$
$\mathrm{GL}_{a+b}(\mathbb{C})$
$\mathrm{GL}_b(\mathbb{C})$
$\mathrm{GL}_c(\mathbb{C})$
$\mathrm{GL}_{a+c}(\mathbb{C})$
$\mathrm{GL}_{b+c}(\mathbb{C})$