Painlevé equations

Dynkin diagrams

and

Hyperkähler manifolds

Philip Boalch (ENS & CNRS)

The Painlevé Equations

$$\mathrm{P_I}: \quad y'' = 6y^2 + t$$

$$\mathrm{P_{II}}: \quad y'' = 2y^3 + ty + \alpha$$

$$\mathrm{P_{III}}: \quad y'' = \frac{(y')^2}{y} - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y}$$

$$\mathrm{P_{IV}}: \quad y'' = \frac{(y')^2}{2y} + \frac{3y^3}{2} + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}$$

$$\mathrm{P_V}: \quad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{t} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{t} - \frac{y(y+1)}{2(y-1)}$$

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where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are parameters.

The Painlevé Equations 4,5,6

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The Painlevé Equations 4,5,6

Okamoto ('80's)

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$\mathbb{C}^2 \circlearrowleft W_{aff}(A_2)$

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The Painlevé Equations 4,5,6

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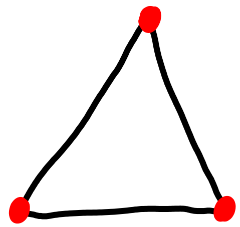
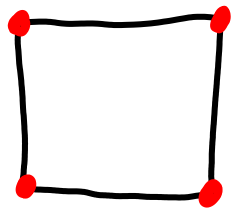
P_{IV}:



$\mathbb{C}^2 \circlearrowleft W_{aff}(A_2)$

P_{V}:



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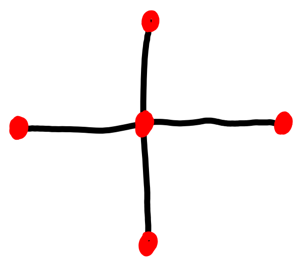
P_{VI}:



$\mathbb{C}^4 \circlearrowleft W_{aff}(D_4)$

Starting point [Hitchin's Frobenius manifold question (1995)]

- Dubrovin (1994) classified semisimple Frobenius manifolds in terms of the "Stokes data" of an operator of form

$$\frac{d}{dz} - \left(\frac{U}{z^2} + \frac{V}{z} \right)$$

U, V $n \times n$ matrices, U diagonal with distinct eigenvalues
V skew-symmetric

e.g $QH^*(\mathbb{CP}^2)$ is a 3d ss Frobenius manifold with Stokes matrix

$$\begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

- Qn Can we obtain a sharp classification in terms of monodromy data of logarithmic connections on $\mathbb{P}^1$?

- Ans Yes — have isomorphism between moduli spaces of such irreg connections & various moduli spaces of log. connections

  (via twisted Fourier–Laplace transform)

  choose good twist to see how to characterise log. conn$^{\text{ns}}$

- Such isoms are braid group equivariant
- & relate natural complex symplectic / Poisson structures

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(Dubrovin 1994)

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• Such isoms are braid group equivariant

• & relate natural complex symplectic / Poisson structures

My (contrary) question: Rather than try to convert questions about irregular connections into the well understood logarithmic / regular singular world, why not try to extend existing technology to the irreg. case?

– Clearly "most" connections on curves are irregular...

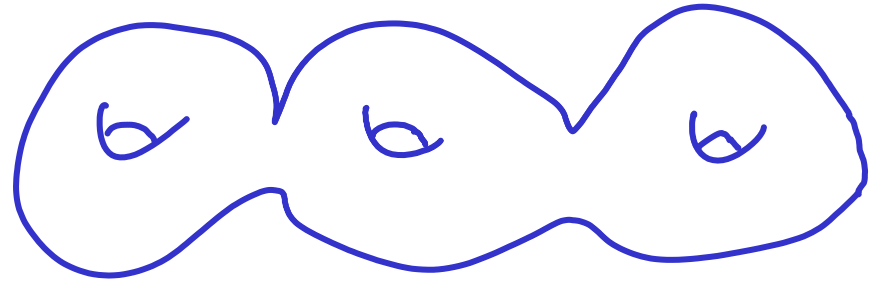
Themes:

- (wild) non-abelian Hodge corresp. on curves
  & hyperkahler moduli spaces

- nonlinear symplectic braid group actions

- understanding such moduli spaces on $\mathbb{P}^1$ $\left(\mathcal{M}^*_{DR} \subset \mathcal{M}_{DR}\right)$

smooth projective curve X



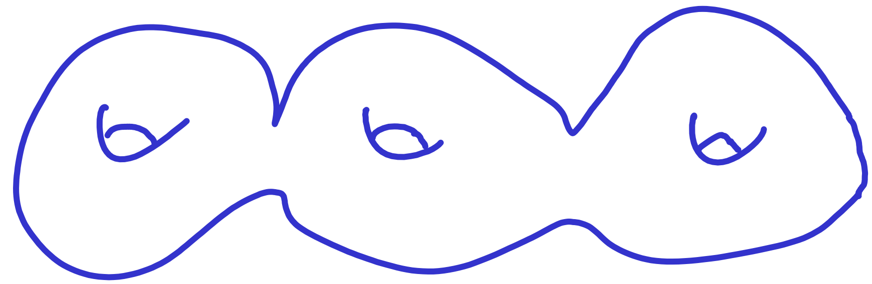
$\mathcal{M} = H^1(X, G)$ $\quad G$ connected complex reductive gp

3 viewpoints :

- $\mathrm{Hom}(\pi_1(X), G)/G$
- G-bundles with holomorphic connections
- G-bundles with Higgs fields

  $\Phi \in H^0(X, \mathrm{ad} P \otimes K)$

smooth projective curve X



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3 viewpoints :

- $\mathrm{Hom}(\pi_1(X), G)/G$ $\quad$ $\mathcal{M}_{Betti}$
- G-bundles with holomorphic connections $\quad$ $\mathcal{M}_{DR}$
- G-bundles with Higgs fields $\quad$ $\mathcal{M}_{Dol}$

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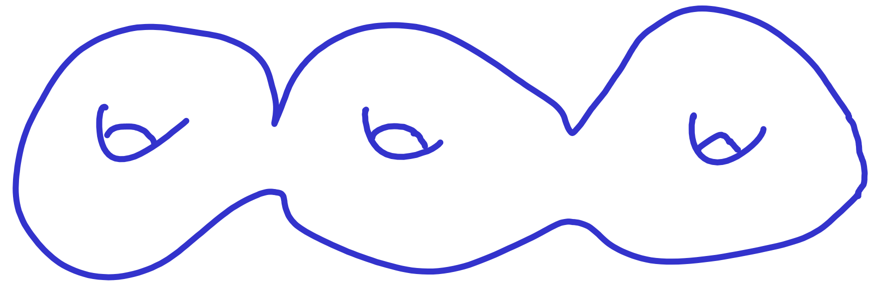
[Add stability conditions (or framings) to get nice moduli spaces]

smooth projective curve X



$\mathcal{M} = H^1(X, G)$ $\quad G$ connected complex reductive gp

- $\mathrm{Hom}(\pi_1(X), G)/G$ $\quad \mathcal{M}_{Betti}$

Riemann-Hilbert $\Updownarrow$

- G-bundles with holomorphic connections $\quad \mathcal{M}_{DR}$

Nonabelian Hodge $\Updownarrow$

- G-bundles with Higgs fields $\quad \mathcal{M}_{Dol}$

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[Add stability conditions (or framings) to get nice moduli spaces]

E.g $G = \mathbb{C}^*$

$$\mathcal{M}_{Dol} = T^*\mathrm{Jac}(X), \quad \mathcal{M}_{Betti} \cong (\mathbb{C}^*)^{2g}, \quad \mathcal{M}_{DR} \longrightarrow \mathrm{Jac}(X)$$

Extend to meromorphic connections $(G = GL_n)$

① Regular singularities $\sim$ open Riemann surfaces

Nonabelian Hodge corresp. (Simpson 1990)

Hyperkahler structures (Nakajima 1996)

Rough picture

- $\mathcal{M}_{Betti} = \mathrm{Hom}(\pi_1(X), G)/G$

fix conjugacy class of monodromy around each puncture to fix symplectic leaves

- $\mathcal{M}_{DR}$ = G-bundles + log connections

- $\mathcal{M}_{Dol}$ = G-bundles + Higgs fields with simple poles

fix adjoint orbits of residues to fix leaves

② Irregular singularities:

Hyperkahler structures & non-abelian Hodge corresp. (Biquard-B. 2004) ⊛

(holom. symplectic structures PB '99) (one direction by Sabbah 1999)

– metrics are complete for generic parameters

Rough picture:

- $\mathcal{M}_{DR}$: fix formal type $(G[\![z]\!])$ to fix symplectic leaves
- $\mathcal{M}_{Betti}$: include Stokes data too
- $\mathcal{M}_{Dol}$: mero. Higgs bundles (ACIHS by Bottacin, Markman 1994)

⊛ only unramified formal types considered – general case is similar (cf. Witten 2007)

Braiding/Mapping class group actions

"Isomonodromy = Nonabelian Gauss-Manin connection"

(extended to irregular case)

Braiding/Mapping class group actions

"Isomonodromy = Nonabelian Gauss-Manin connection"

(PB 2001)

(extended to irregular case)

1900's (higher) Painlevé equations

~ 1980 Sato Miwa Jimbo Ueno ...

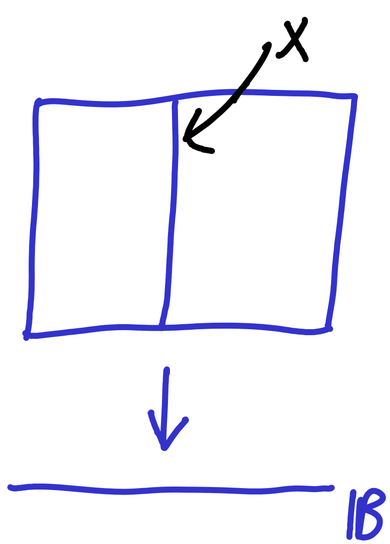
Simpson 1994

Braiding/Mapping class group actions

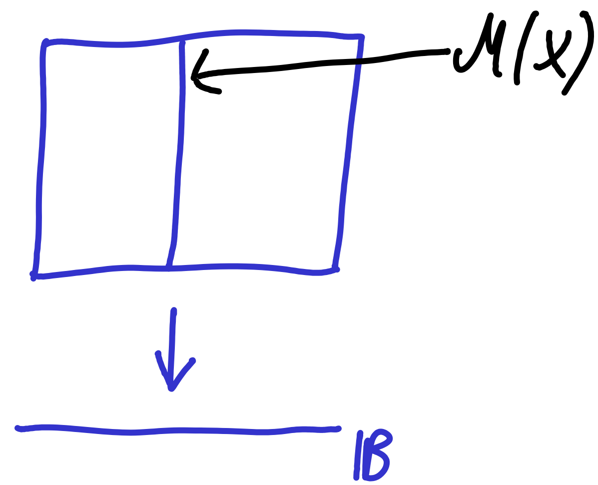
"Isomonodromy = Nonabelian Gauss-Manin connection"

(extended to irregular case)

Reg. case:



$\Longrightarrow$



family of curves with marked points

'family' of moduli spaces

- nonlinear fibre bundle with flat algebraic connection
- Betti spaces form a local system of varieties

What is the base $\mathbb{B}$ in the irregular case?

Def^{n} Fix $T \subset G$

An "irregular curve" is

- A smooth complex curve Σ, with
- distinct marked points $a_1, \dots, a_m$, and
- an irregular type Q at each marked point

$$Q = \frac{A_r}{z^r} + \cdots + \frac{A_1}{z} \in \mathfrak{t}((z)) / \mathfrak{t}[[z]]$$

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Look at admissible deformations, such that:

- Σ remains smooth,
- points a_i remain distinct
- Pole Order $(\alpha \circ Q)$ does not change $(\forall$ roots $\alpha \in R \subset \mathfrak{t}^*)$

e.g. if $A_r \in \mathfrak{t}_{reg}$

Basic fact:

If we consider meromorphic connections A on G-bundles $/\Sigma$ with poles at $\{a_i\}$ s.t. locally:

$$A \underset{G[\![z]\!]}{\sim} dQ + \Lambda \frac{dz}{z} \qquad \text{at each pole}$$

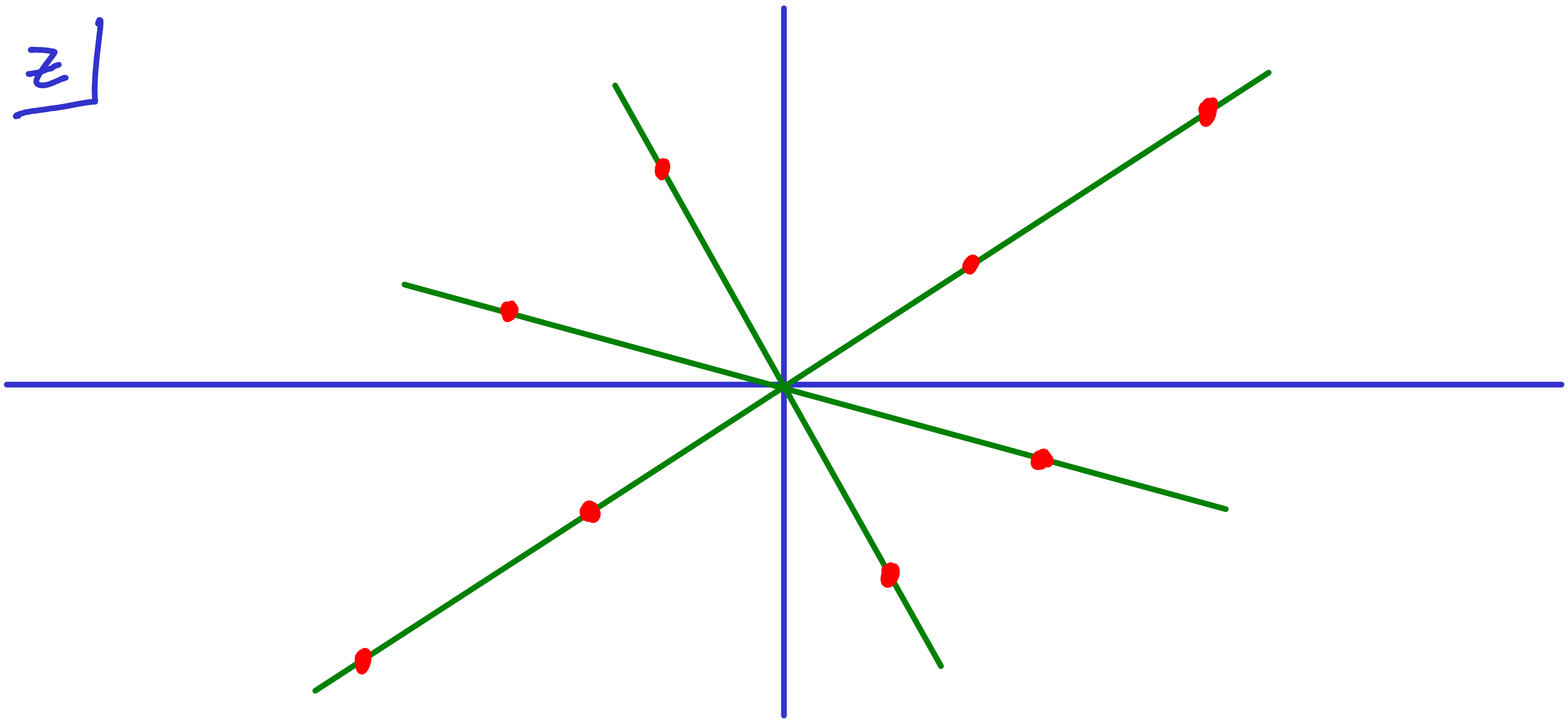
holom.

Then again get flat algebraic con^n on family of moduli spaces over the space of (irregular) curve deformations

cf. Jimbo-Miwa-Ueno '81 (GL_n), PB '02 (other G)

Simplest example (PB '02) $\quad r=1, \quad Q = \frac{-A_1}{z}, \quad A_1 \in t_{reg}$

Plot roots on z-plane: $\langle A_1, R \rangle \subset \mathbb{C}^*$



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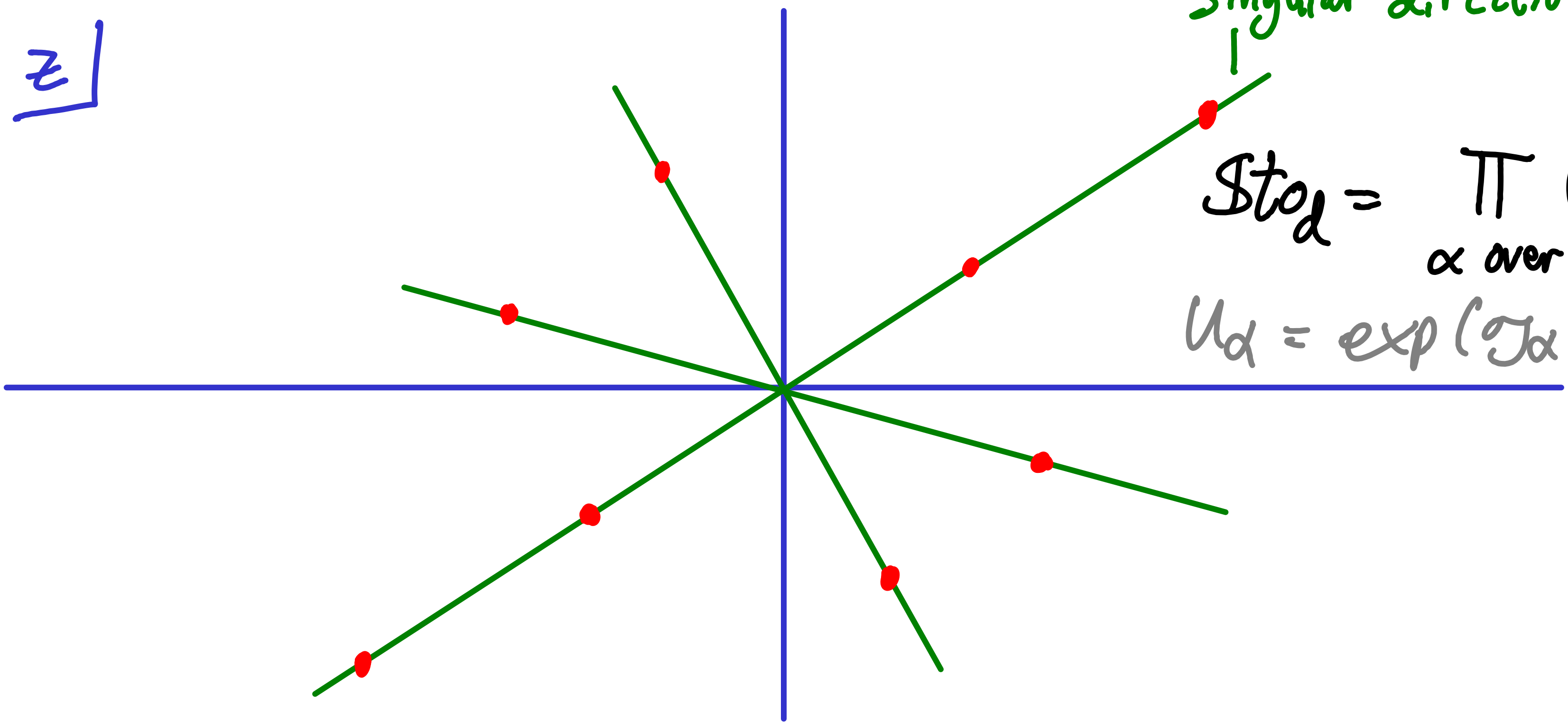
singular directions d



$$\mathbb{S}to_d = \prod_{\alpha \text{ over } d} U_\alpha \subset G$$

$U_\alpha = \exp(\mathfrak{g}_\alpha)$ root gp

$$\{\text{Stokes data}\} = \prod_d \mathbb{S}to_d$$

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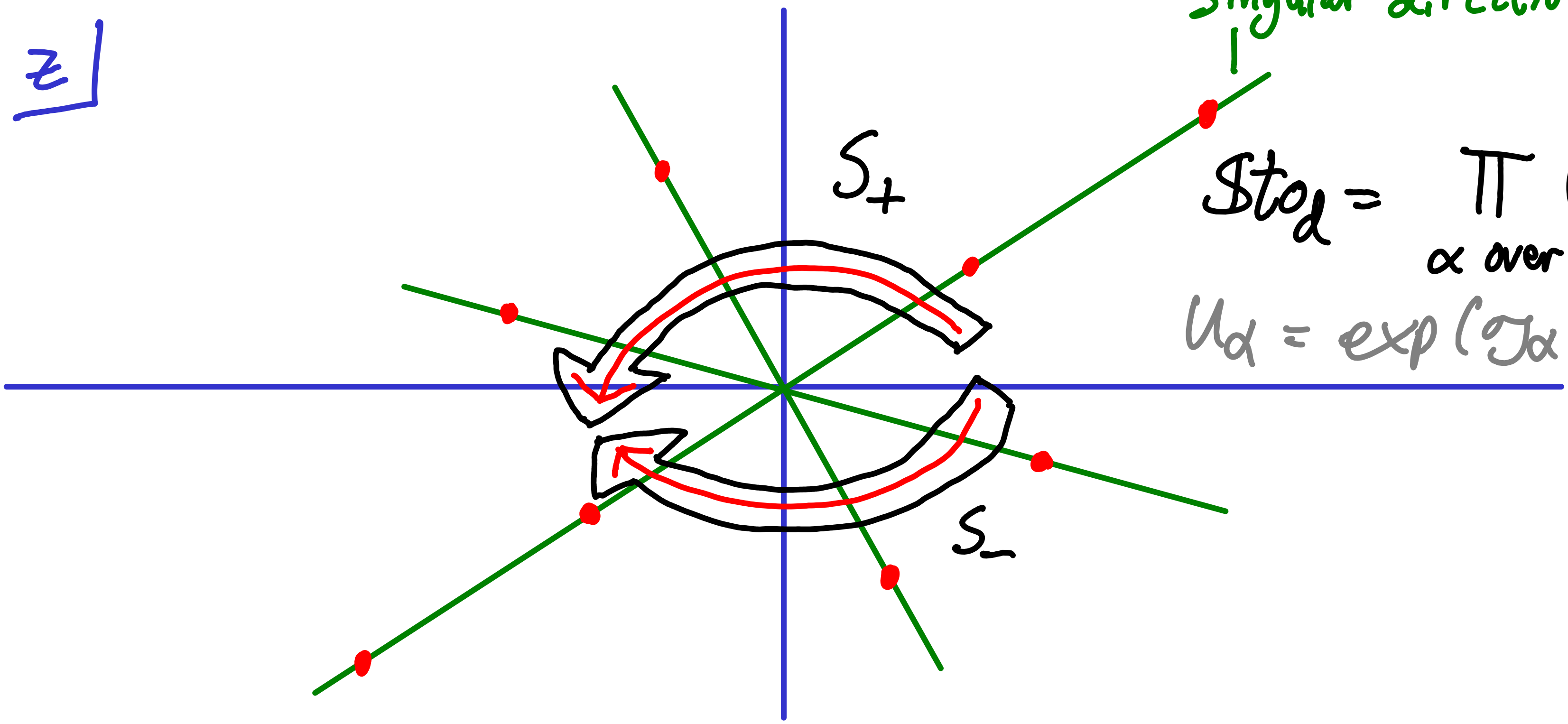

$\mathrm{Sto}_d = \prod_{\alpha \text{ over } d} U_\alpha \subset G$

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$\{\text{Stokes data}\} = \prod_d \mathrm{Sto}_d \cong \underbrace{U_+ \times U_-}_{\text{unipotent radicals of opposite Borels}} \ni (S_+, S_-)$

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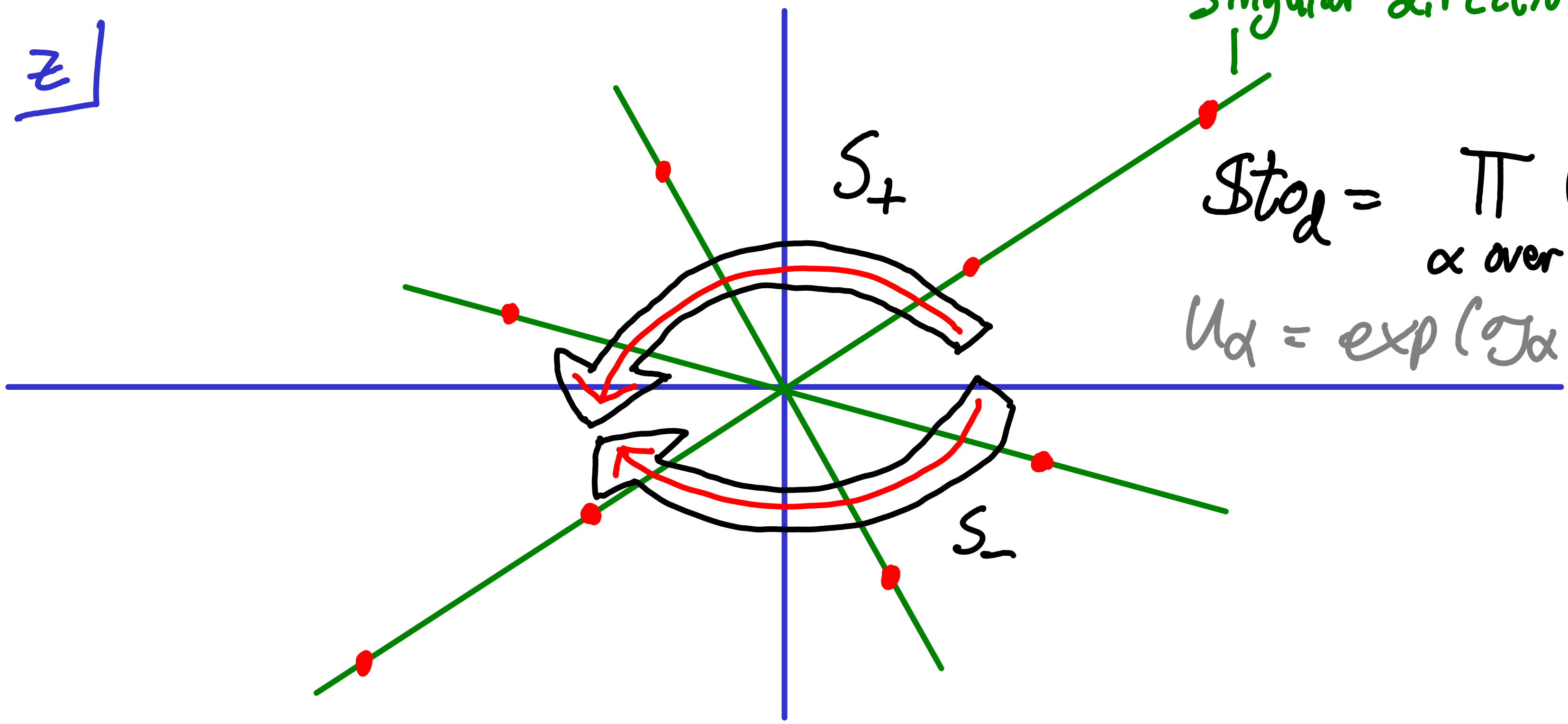

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Isomonodromy: Vary $A_1 \in \mathfrak{t}_{reg}$ & keep $S_\pm$ const. (locally)

In this example the resulting braid gp action had been previously seen:

$$\begin{tikzcd}
 & U_q\mathfrak{g} & \\
\mathrm{Fun}(G^*) \arrow[ur] & & U\mathfrak{g} \arrow[ul] \\
 & \mathrm{Fun}(\mathfrak{g}^*) = \mathrm{Sym}(\mathfrak{g}) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$G^* \cong T \ltimes (U_+ \times U_-)$ dual Poisson-Lie group (Drinfeld)

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$B_{\mathfrak{g}} \circlearrowright U_q\mathfrak{g}$ Quantum Weyl group action (Lusztig, Soibelman ...)

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Quantum Weyl group action
(Lusztig, Soibelman ...)

$G^* \cong T \ltimes (U_+ \times U_-)$ dual Poisson-Lie group (Drinfeld)

Thm (- '02)

The DKP action arises from isomonodromy ($U_+ \times U_-$ = Stokes data)
- Purely geometric origin (not just explicit generators)
- $U_q\mathfrak{g}$ thus quantizes a moduli space of meromorphic connections

Example (cont.)

$$B \in \mathfrak{g}^*$$

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Given $A_1 \in \mathfrak{t}_{reg}$

$$\left(\frac{A_1}{z^2} + \frac{B}{z}\right) dz$$

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$$B \in \mathfrak{g}^*$$

Given $A_1 \in \mathfrak{t}_{reg}$

$$\left(\frac{A_1}{z^2} + \frac{B}{z}\right) dz \longrightarrow \text{Stokes data}$$

Example (cont.)

$$B \in \mathfrak{g}^* \xrightarrow{\nu_{A_1}} G^*$$

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Thm (PB '01–'02)

ν_{A_1} is Poisson & is generically a local analytic isomorphism

($\Rightarrow$ new direct proofs certain results of Duistermaat, Ginzburg–Weinstein, Kostant)

Example (cont.)

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Given $A_1 \in \mathfrak{t}_{reg}$ $\quad \left(\frac{A_1}{z^2} + \frac{B}{z}\right)dz \longrightarrow$ Stokes data

Thm (PB '01-'02)

ν_{A_1} is Poisson & is generically a local analytic isomorphism

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Isomonodromy equations: $\quad dB = [B, \mathrm{ad}_{A_1}^{-1}[dA_1, B]]$

($\mathfrak{t}_{reg} = \{A_1\} =$ 'times')

Formula for (part of) ν_{A_1} by Bridgeland–Toledano ~ 2008

Summary

Irregular curve

$\Downarrow$

Hyperkaehler manifold

$\mathcal{M}$

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Irregular curve (+ weighted conjugacy classes)

$$\Downarrow$$

Hyperkahler manifold

$\mathcal{M}$

(wild nonabelian Hodge structure)

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Irregular curve (+ weighted conjugacy classes)

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$\Downarrow$

$\mathcal{M}_{Dol}$	$\mathcal{M}_{DR}$	$\mathcal{M}_B$
Higgs bundles	mero. connections	monodromy & Stokes data
algebraic integrable systems (mero. Hitchin systems)	isomonodromy systems	symplectic braid & (irregular) mapping class group actions

Guide to moduli spaces on $\mathbb{P}^1$

Typically $\qquad \mathcal{M}^* \subset \mathcal{M}$

open part where bundle holom. trivial $/\mathbb{P}^1$

& $\mathcal{M}^*$ again a complete hyperkahler manifold

"approximation" to more transcendental metric on $\mathcal{M}$

Classical hyperkahler mfds

1. Complex coadjoint orbits $\quad \mathcal{O} \subset \mathfrak{g}^*$

(Kronheimer, Biquard, Kovalev)

If pole divisor $2(0) + (\infty) \subset \mathbb{P}^1$

have examples where $\quad \mathcal{M}^* \cong \mathcal{O} /\!\!/\!\!/_{\lambda} T_K$

$\left[\mathcal{M}_{Betti} = \mathcal{L} /\!\!/_{\lambda} T, \quad \mathcal{L} \subset G^* \text{ symplectic leaf} \right]$

($T_K \subset T$ compact torus)

2 T^*G (Kronheimer)

If pole divisor $2(0) + 2(\infty) \subset \mathbb{P}^1$

have examples where $\mathcal{M}^* \cong T_K \backslash\!\backslash\!\backslash_{\lambda_1} T^*G /\!/\!/_{\lambda_2} T_K$

$$\left[\begin{array}{l} \mathcal{M}_{Betti} = T \backslash\!\backslash_{\lambda_1} \mathbb{D} /\!/_{\lambda_2} T \\ \mathbb{D} \subset (G \times G^*)^2 \quad \text{Lu-Weinstein double sympl. groupoid} \end{array}\right]$$

③ ALE spaces $\quad$ deformations of $\widetilde{\mathbb{C}^2/\Gamma}$

(Eguchi–Hanson, Gibbons–Hawking, Hitchin, Kronheimer)

$\dim_{\mathbb{R}} = 4$ $\quad$ (gravitational instantons / quaternionic curves)

$\Gamma \subset SU_2$ finite $\longleftrightarrow$ ADE affine Dynkin graph

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Fact In cases E_8, E_7, E_6, D_4 (logarithmic realisations), A_3, A_2, A_1 (only irregular realisations)

have $\mathcal{M}$ s.t $\mathcal{M}^* \subset \mathcal{M}$ is corresponding ALE space

	Pole orders
A_3	2 +1 +1
A_2	3 + 1
A_1	4

– Okamoto found in 1987 the corresponding affine Weyl groups are the sym gps of the corresponding Painlevé equations

Rough classification (of CM's) in $\dim_{\mathbb{C}} = 2$:

E_8 E_7 E_6

D_4 $A_3 = D_3$ D_2 D_1 (D_0)

A_2 A_1 (A_0)

reg $\leftarrow$ | $\rightarrow$ irreg

④ (Nakajima) Quiver varieties

$$\overset{V}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{W}{\circ}$$

$\mathrm{Hom}(V,W) \oplus \mathrm{Hom}(W,V)$ is hyperkahler $U(V) \times U(W)$ space

Graph = ADE dynkin graph $\Rightarrow$ ALE space (Kronheimer)
else in general get higher dim^n hyperkahler mfd (or empty)

— lets consider simply-laced cases

E.g. Fuchsian case $\quad G = GL_n(\mathbb{C})$

$$\mathcal{M}^* \cong \mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!\!/ G$$

$(\mathcal{O}_i \subset \mathfrak{g}^*$ coadjoint orbits$)$

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point of $\mathcal{M}^*$ $\sim$ Fuchsian system $\displaystyle\sum_1^m \frac{A_i}{z - a_i}\, dz$ $\quad A_i \in \mathcal{O}_i$

$\sum A_i = 0$

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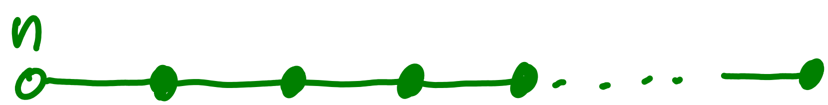
Relation to quivers (Kraft-Procesi, Nakajima, ...)

$\mathcal{O}_i \cong$



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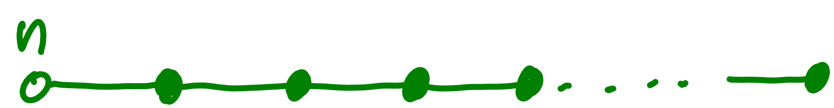
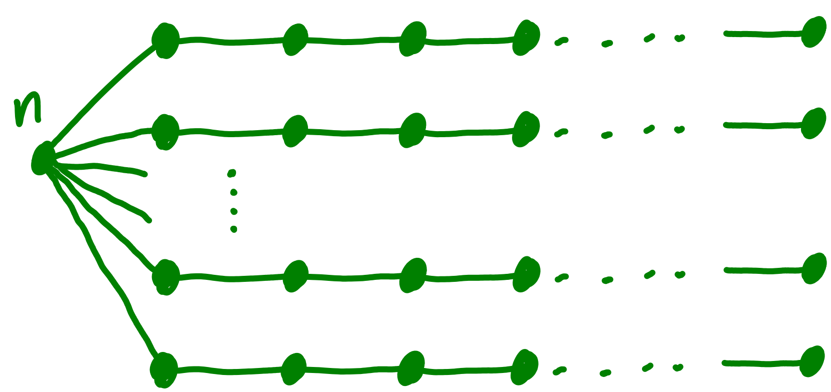
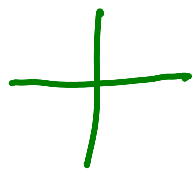
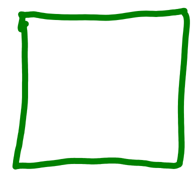
$\mathcal{O}_i \cong$ 

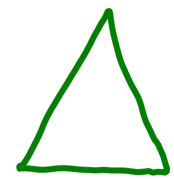
$\mathcal{M}^* \cong$ 

"Starshaped" quivers used by Crawley-Boevey in Deligne-Simpson problem

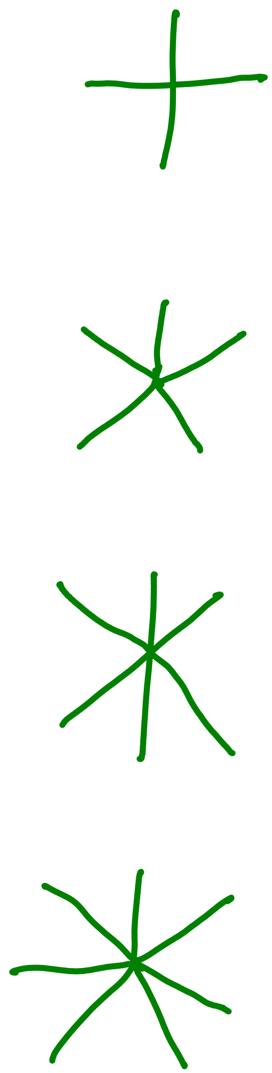
Recall Okamoto showed the Painlevé equations 4, 5, 6 have affine Weyl group symmetries of type A_2, A_3, D_4 resp.

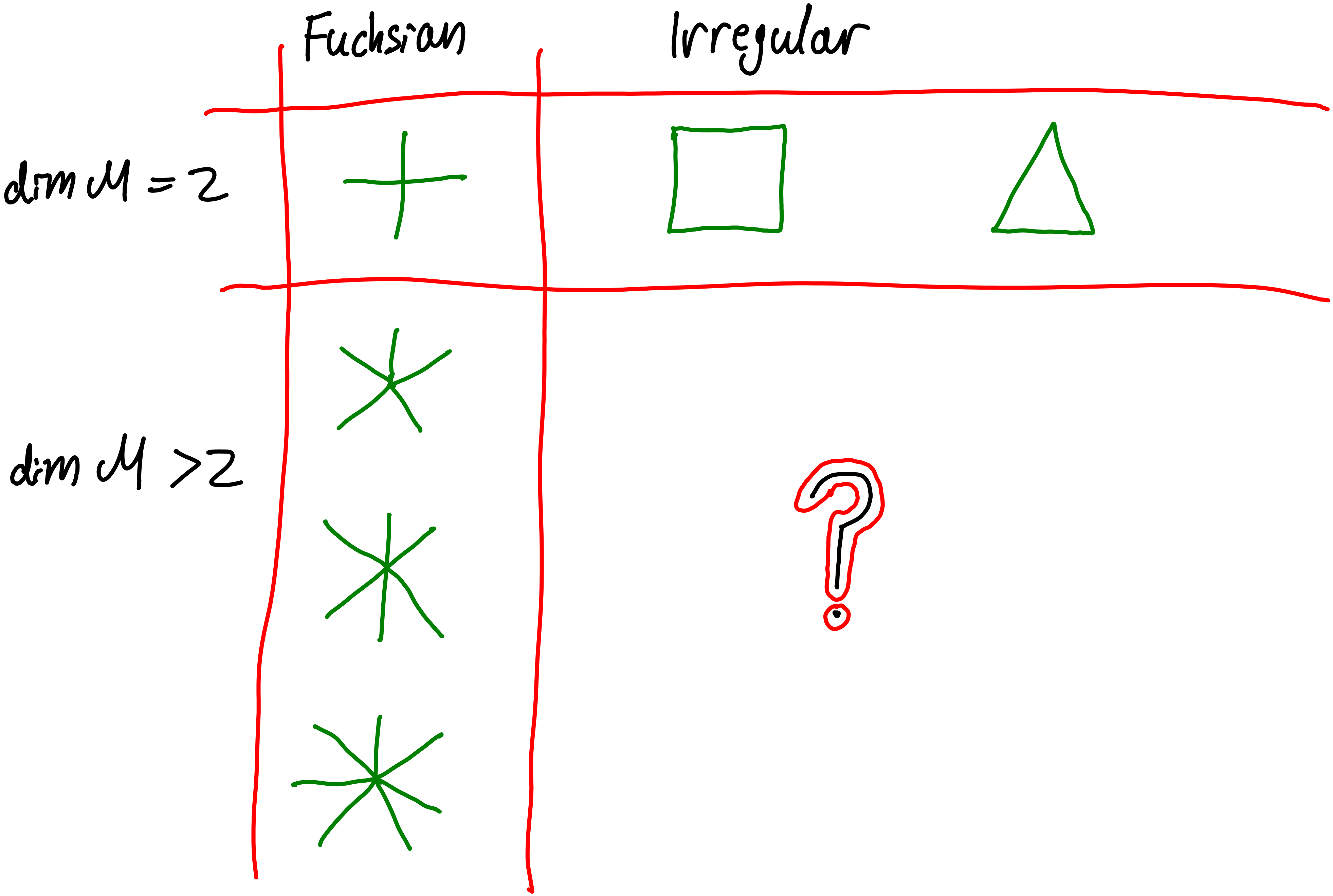






Recall Crawley-Boevey related moduli spaces of Fuchsian systems to star-shaped quivers (building on Kraft-Procesi, Nakajima, ...)

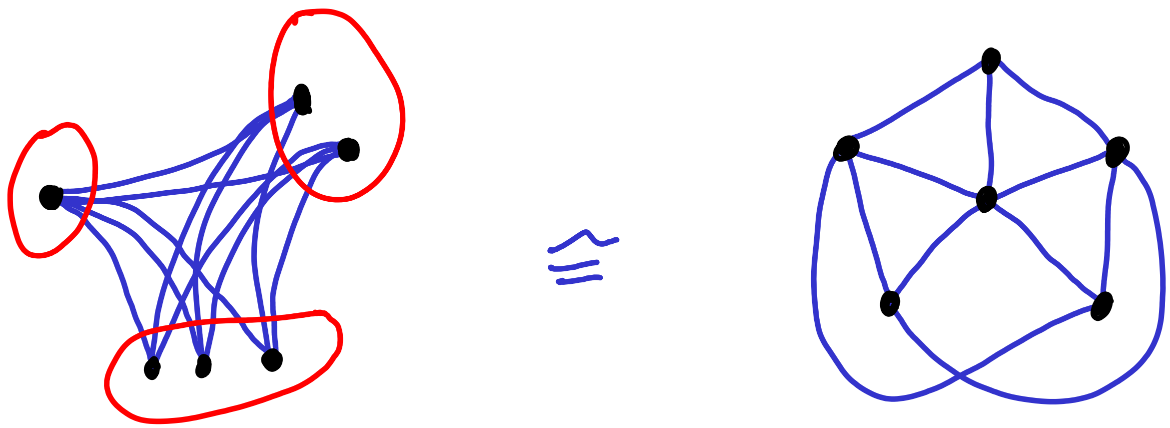



Fuchsian
Irregular
dim $\mathcal{M} = 2$
dim $\mathcal{M} > 2$
?

Thm

Can take any complete k-partite graph (for any k)

E.g.



$\Gamma(3,2,1)$

- get action of corresponding (not necessarily affine) Kac-Moody Weyl group

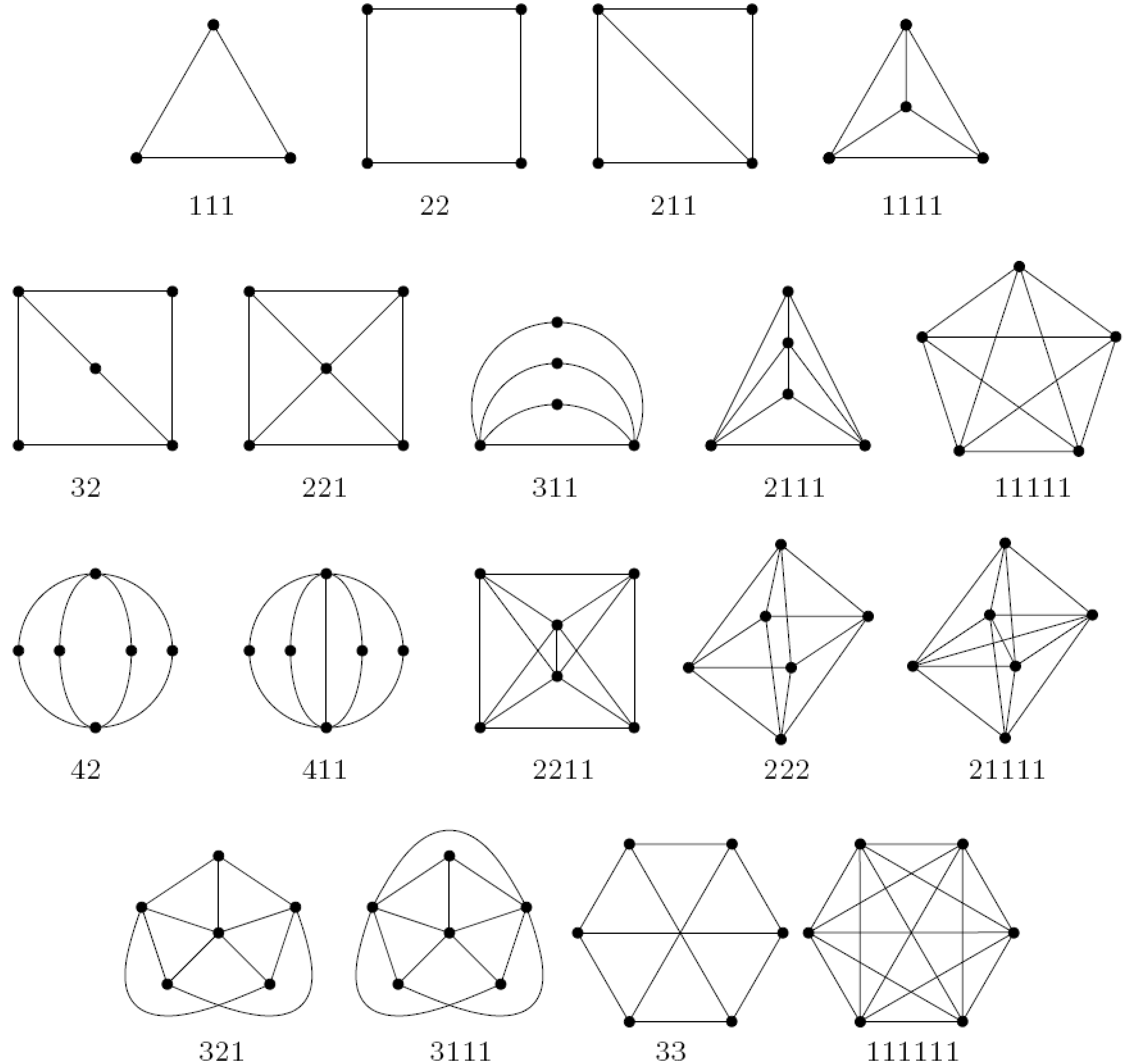


FIGURE 1. Graphs from partitions of $N \leq 6$
(omitting the stars $\Gamma(n, 1)$ and the totally disconnected graphs $\Gamma(n)$)

More general and precise statement:

Definition A graph Γ is a

- "nonabelian Hodge graph" if there is some (rational) irregular curve Σ

s.t. $$\begin{array}{c} \mathcal{M}^*(\Sigma) \cong \text{a quiver variety attached to } \Gamma \\ \cap \\ \mathcal{M}(\Sigma) \end{array}$$

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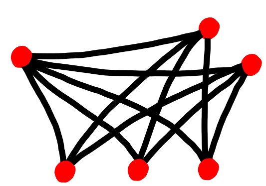
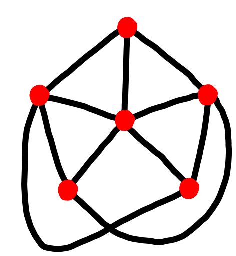


$\cong$



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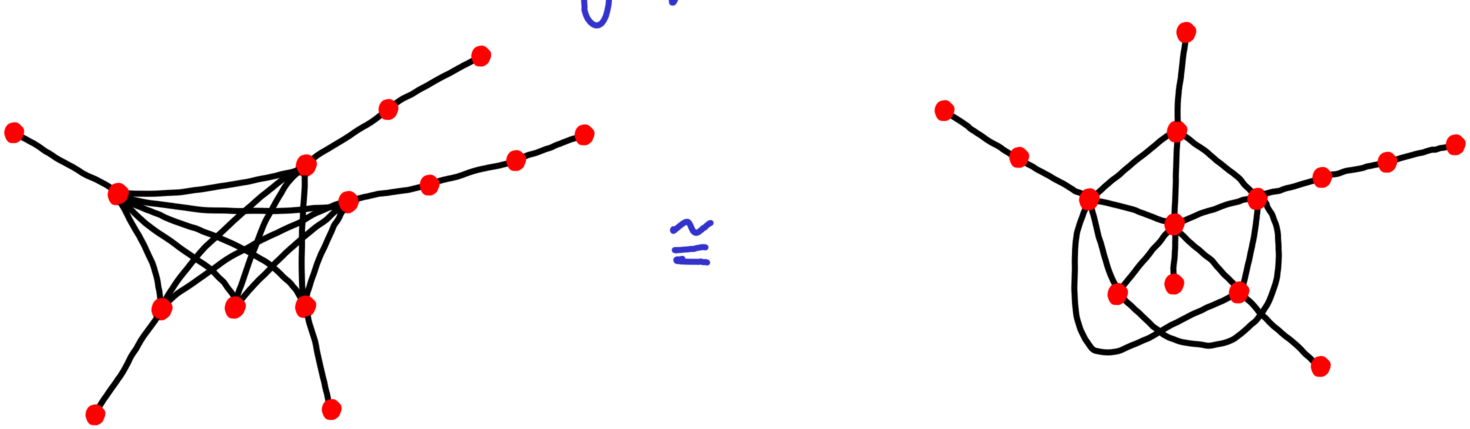

— generalising the star-shaped graphs

Thm

Any supernova graph is a nonabelian Hodge graph

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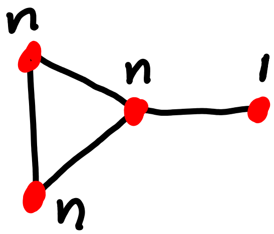
Moreover Γ determines a (symmetric) Kac-Moody root system & Weyl group, and Weyl group elements lift to give isomorphisms between such systems

E.g. Higher/hyperbolic/Hilbert Painlevé systems

$\Gamma_n =$ 

$\Longrightarrow \quad hP^n_{IV} := \mathcal{M}(\Gamma_n)$ dimension $2n$

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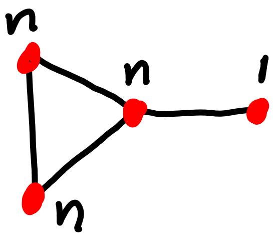
$\Gamma_n =$ 

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$n=1 \qquad hP^1_{IV} \cong P_{IV} \quad$ dim 2

$$\mathcal{M}^*(\Gamma_n) \underset{\text{diffeo}}{\cong} \mathrm{Hilb}^n(\mathcal{M}^*(\Gamma_1))$$

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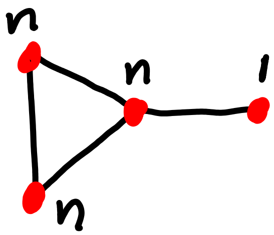
$\Gamma_n =$ 

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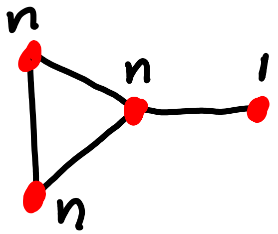
$\Gamma_n =$ 

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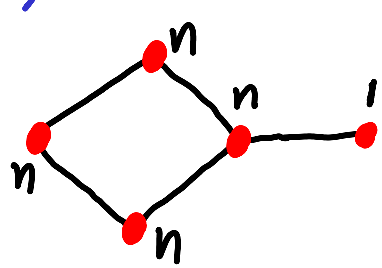
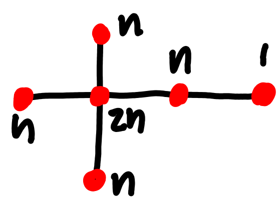
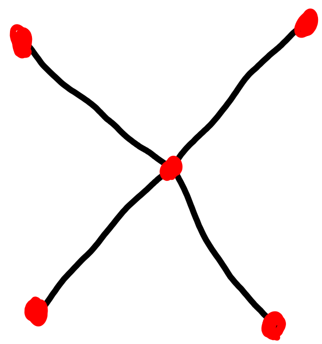
similarly for any 2d Hitchin system e.g:

$\Gamma_n =$ 

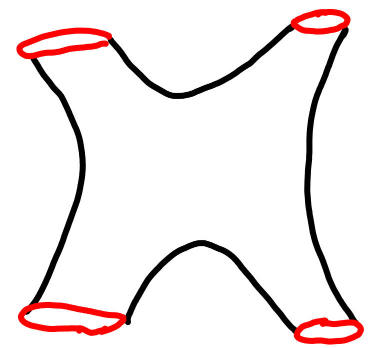
$\Longrightarrow \quad hP^n_{V} := \mathcal{M}(\Gamma_n)$ dimension $2n$

$\Gamma_n =$ 

$\Longrightarrow \quad hP^n_{VI} := \mathcal{M}(\Gamma_n)$ dimension $2n$



$\mathcal{M}^*$

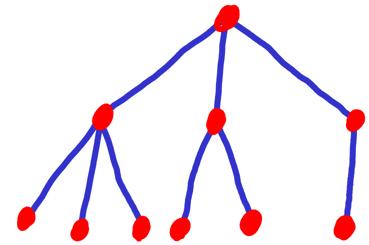


$\mathcal{M}$

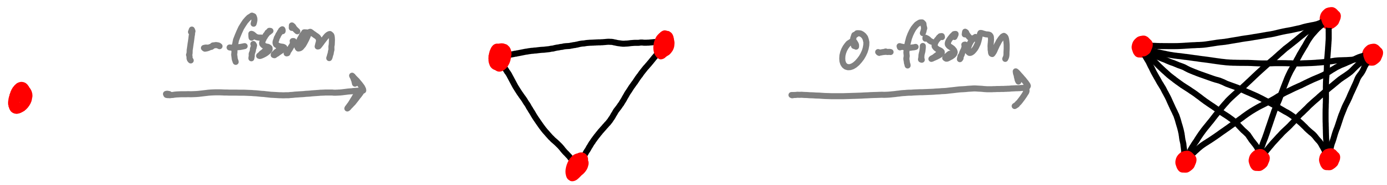
Fission picture

Partitions $\longleftrightarrow$ Height 3 rooted trees (fission tree)

$3+2+1$



$\longleftrightarrow$ complete k-partite graphs



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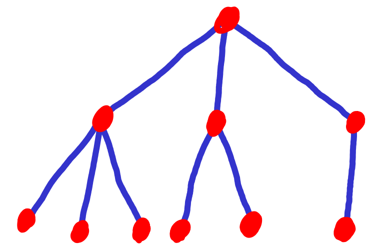

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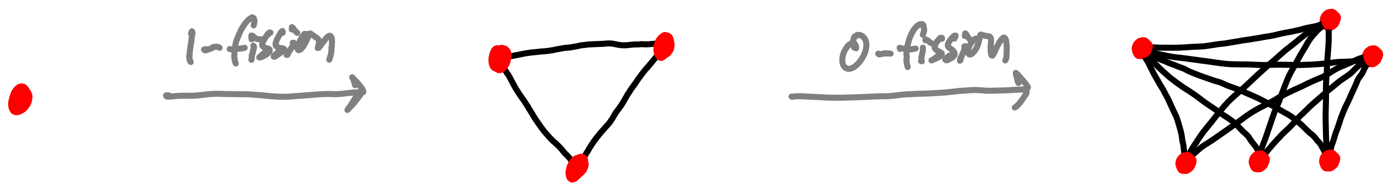
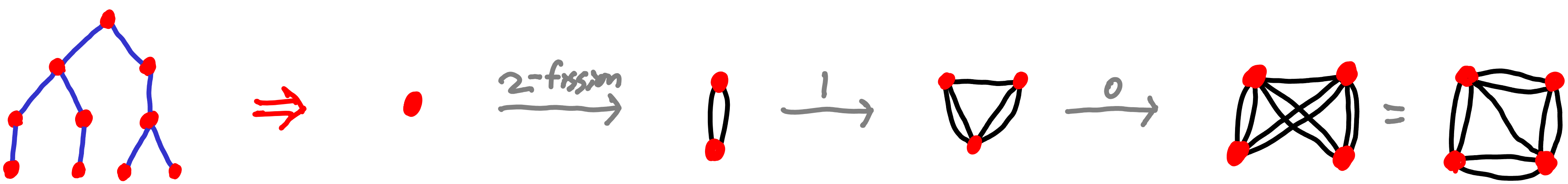


More generally can have 'higher' trees and multiple edges:



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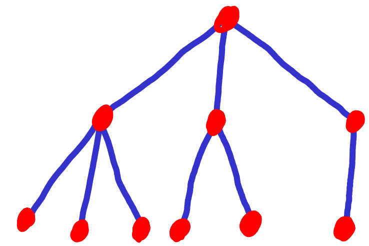

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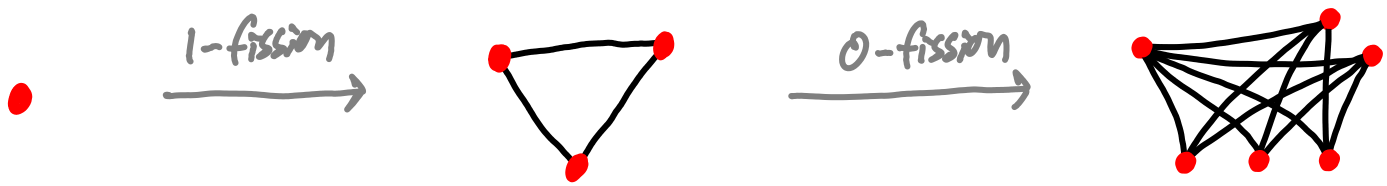

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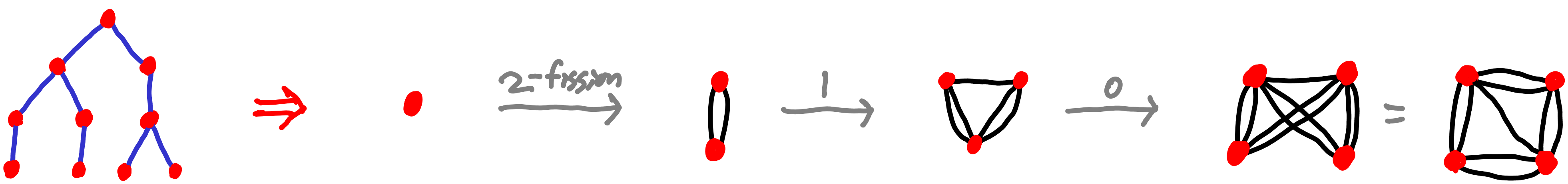

E.g.

$\Rightarrow \mathcal{M}^* \cong T^*\mathbb{CP}^{n-1}$

Calabi's examples