Transformation groups for Isomonodromy equations

Nov. 2012
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Most loved examples

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1 Painlevé equations

P_{VI} , P_V , P_{IV} , P_{III} , P_{II} , P_I

$$\left(\frac{d}{dt}\right)^2 y(t) = \cdots$$

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① Painlevé equations

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② Schlesinger's equations

$$dA_i = -\sum_{j\neq i} [A_i, A_j]\, d\log(t_i - t_j)$$

$$\underset{\sim}{t} \in \mathbb{B} = \mathbb{C}^m \setminus \text{diags}, \qquad A_i(\underset{\sim}{t}) \in \mathfrak{gl}_n(\mathbb{C})$$

Two types of discrete groups appear

1. Weyl groups

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① Weyl groups

Okamoto:

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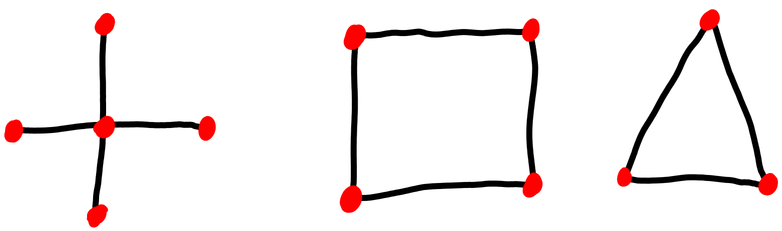
$D_4^{(1)}$	$A_3^{(1)}$	$A_2^{(1)}$	$D_2^{(1)}$	$A_1^{(1)}$	$A_0^{(1)}$



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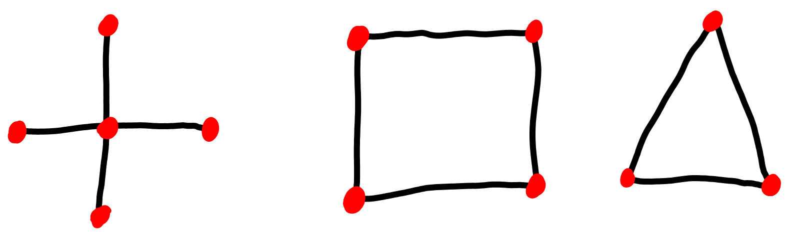

. . . .

Geometry not immediately clear

$D_4^{(1)} \sim L\,SO_8$, but $P_{VI} \sim$ IMDs of rank 2 connections with four poles on $\mathbb{P}^1$

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2 Braid/mapping class groups (nonlinear monodromy)

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$\pi_1(\mathbb{B})$ = pure braid group on m strands $= P_m$

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2. Braid / mapping class groups (nonlinear monodromy)

For Schlesinger's equations:

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Malgrange/Miwa (~80): Linear connection $\sum_1^m \frac{A_i}{z-t_i} dz$ on $\mathbb{P}^1$

$\Downarrow$

Meromorphic solution of Schlesinger system on $\widetilde{\mathbb{B}}$ $\left(\mathbb{B} = \widetilde{\mathbb{B}}/P_m\right)$

E.g. P_m: $m=3$ $P_3 \longrightarrow P_3/\mathbb{Z} = \mathrm{Free}_2 \cong \Gamma(2) \subset PSL_2(\mathbb{Z})$

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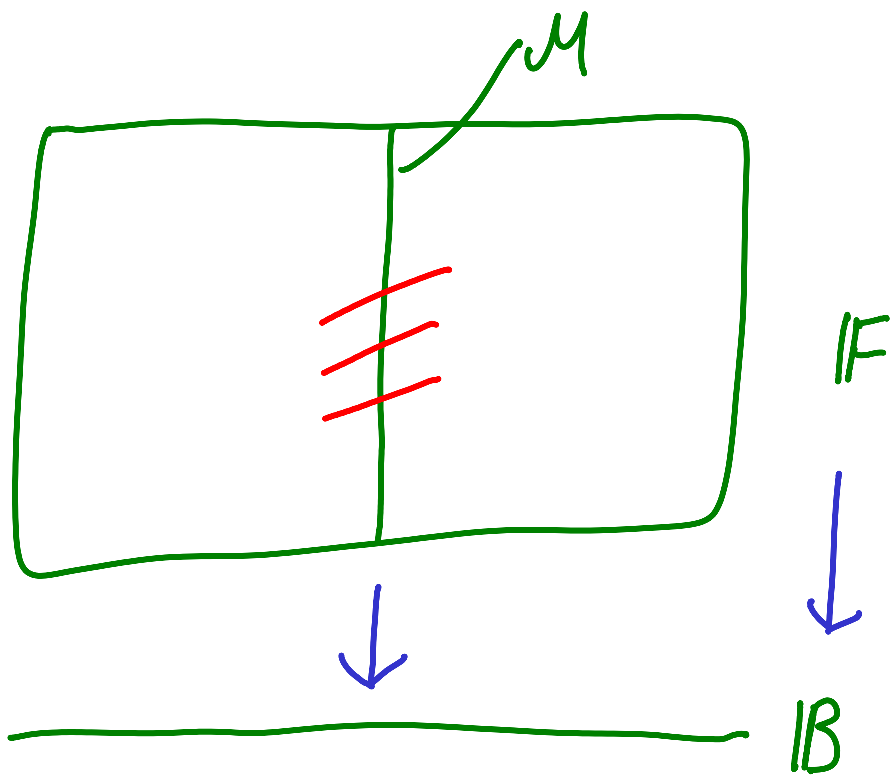
Sketch/Idea

1. Rephrase isomonodromy equations as nonlinear connections on fibre bundles



(cf. PB '99, '01)

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1. Rephrase isomonodromy equations as nonlinear connections on fibre bundles

2. Let $\mathbb{B}$ be the parameter space of an "admissible family of irregular curves"

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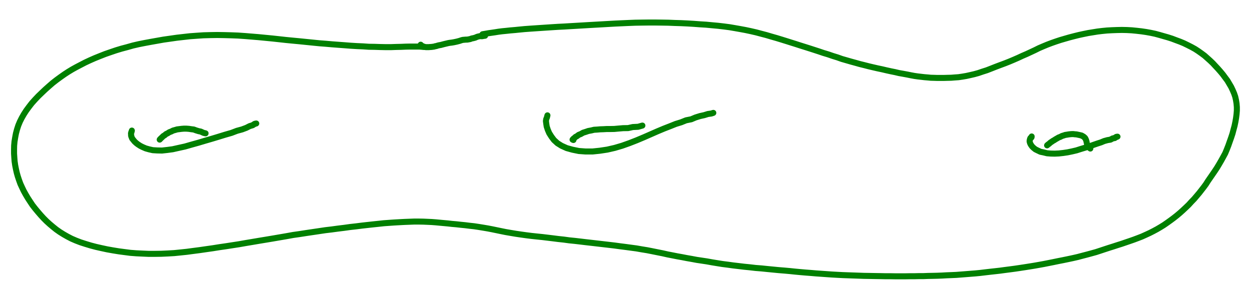
$$\begin{array}{ccc} \Sigma_b & \subset & \Sigma \\ & & \downarrow \\ & & \mathbb{B} \end{array}$$

Smooth Riemann surface



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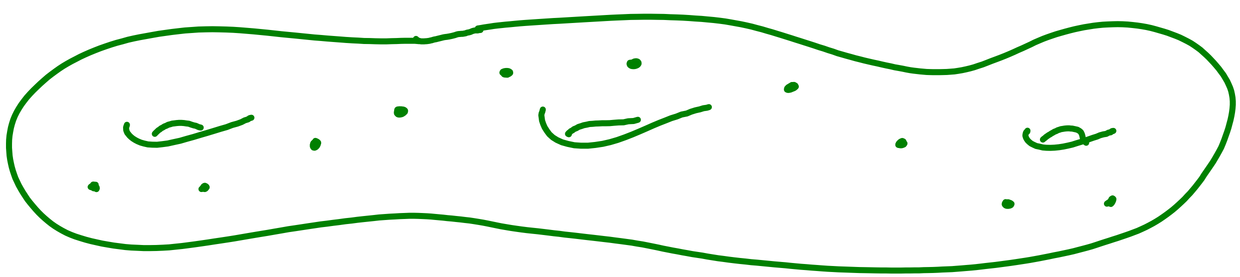

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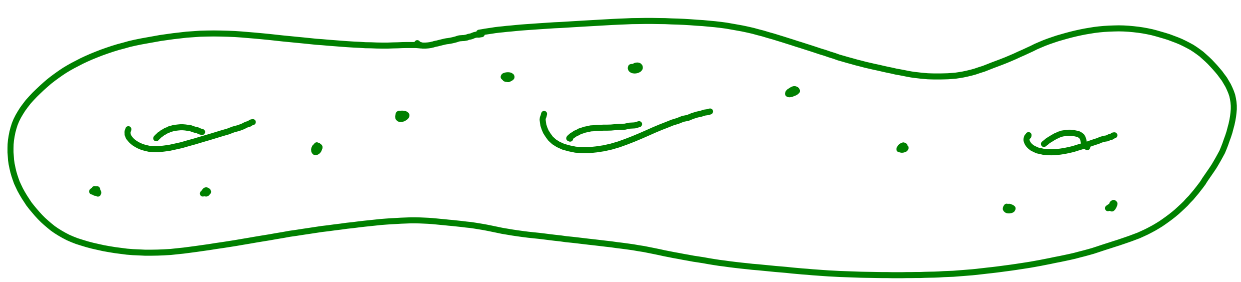

+ marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$

+ an "irregular type" $Q_i = \dfrac{A_{r_i}}{z^{r_i}} + \dots + \dfrac{A_1}{z}$ at each a_i $\left\{ \begin{array}{l} A_i \in \mathfrak{t} \\ z(a_i) = 0 \end{array} \right.$

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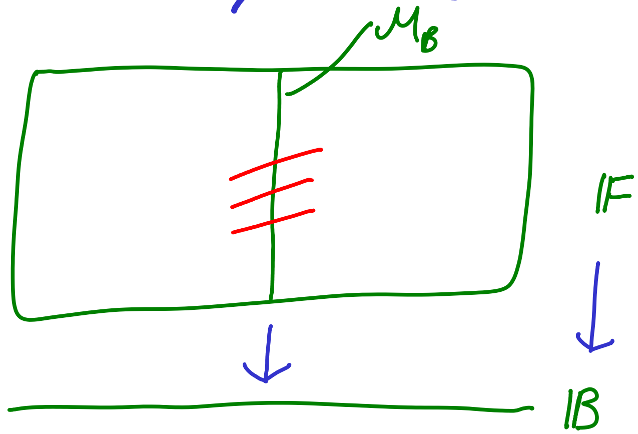

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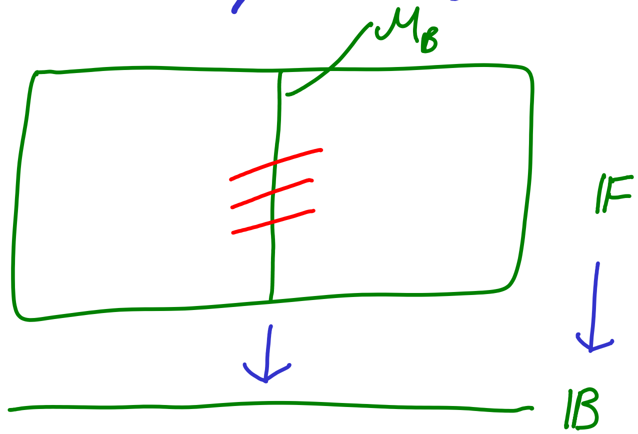

$\Rightarrow$ integrates to give nonlinear monodromy action

$$\boxed{\pi_1(\mathbb{B}) \circlearrowleft \mathcal{M}_B(\Sigma_b)}$$

(generalises Hurwitz braid action & q-c q Weyl)

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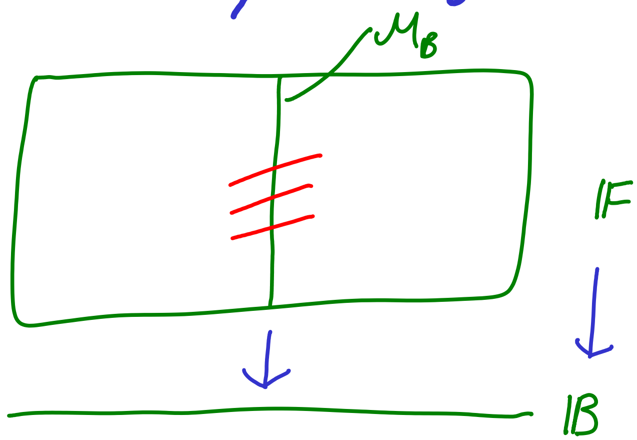

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⑤ Conjugate by Riemann-Hilbert-Birkhoff to get isomonodromy connection

$$\begin{array}{ccccccc}
\mathcal{M}_{DR}(\Sigma_b) & \hookrightarrow & \mathbb{F}_{DR} & \xrightarrow[\cong]{RHB} & \mathbb{F}_B & \hookleftarrow & \mathcal{M}_B(\Sigma_b) \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbb{B} & = & \mathbb{B} & &
\end{array}$$

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Three levels: 1 Nonlinear connection

2 Matrix equations (e.g. Schlesinger, JMMS, JMU)

3 Scalar equations (e.g. Painlevé equations)

— each has strengths and weaknesses

	S	W
1	coord indep't.	hard to teach to undergraduates
2	good balance explicit vs generality	(can get complicated)
3	as explicit as possible	many equivalent expressions not known/too messy in many cases

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- e.g. JMU's injective RHB map is $\mathcal{M}^* \hookrightarrow \mathcal{M}_B$

& it factors as $\mathcal{M}^* \subset \mathcal{M}_{DR} \xrightarrow[\text{RHB}]{\cong} \mathcal{M}_B$ (P.B '99, '01)

Weyl group transformations

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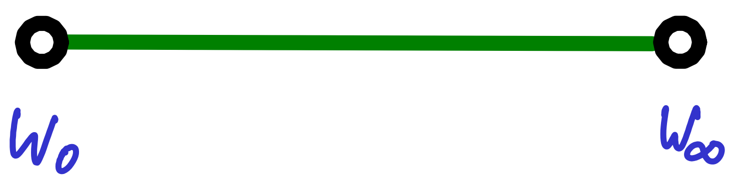
Revisit the JMMS equations (Jimbo-Miwa-Mori-Sato 1980)

Choose two finite dimensional vector spaces W_0, W_∞



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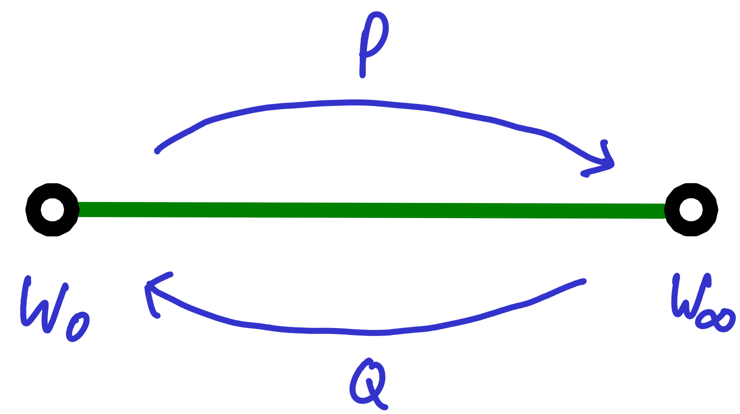

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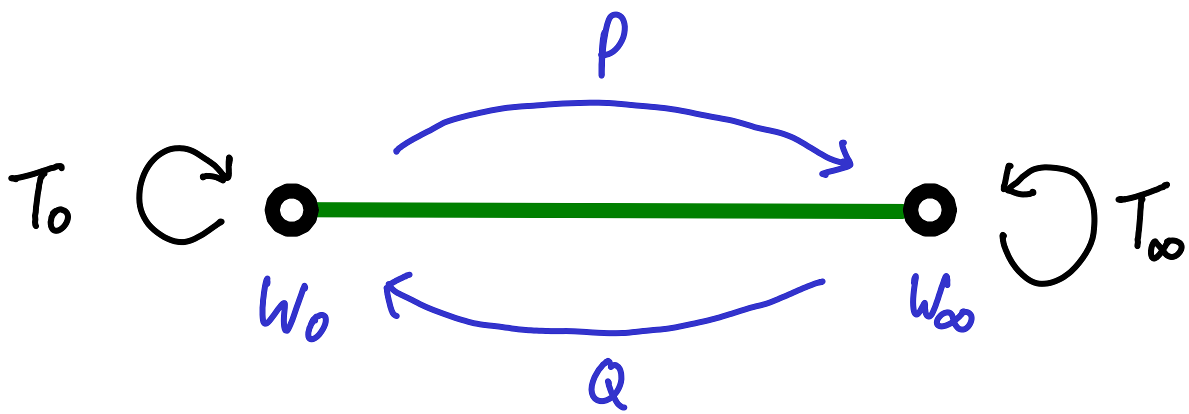
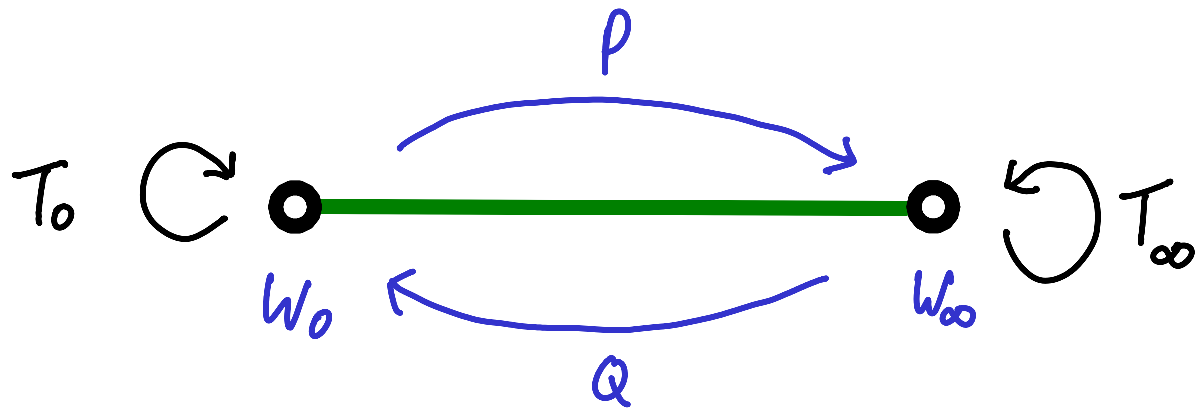


T_i diagonalisable; Eigenvalues of T_i are the times

— no further coalescences permitted

— eigen spaces fixed



Can write JMMS equations as follows:

$$dQ = Q\,\widetilde{PQ} + \widetilde{QP}\,Q + T_0\,Q\,dT_\infty + dT_0\,Q\,T_\infty$$

$$-dP = P\,\widetilde{QP} + \widetilde{PQ}\,P + T_\infty\,P\,dT_0 + dT_\infty\,P\,T_0$$

$$
\begin{tikzcd}
W_0 \arrow[loop left, "T_0"] \arrow[r, dash] \arrow[r, bend left=50, "P"] & W_\infty \arrow[loop right, "T_\infty"] \arrow[l, bend left=50, "Q"]
\end{tikzcd}
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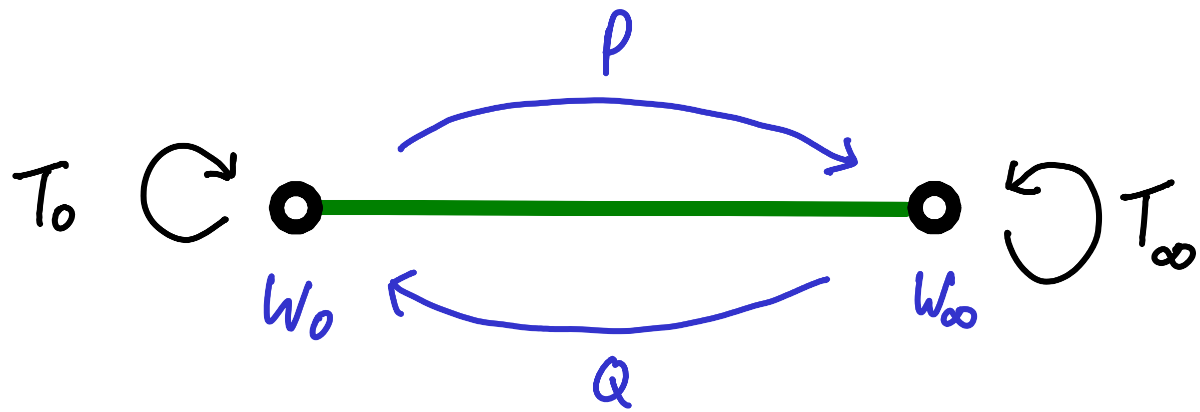
where $\tilde{R} = \mathrm{ad}_{T_i}^{-1}[dT_i, R]$ for $R \in \mathrm{End}(W_i)$

$$\left(\tilde{R}_{ab} = R_{ab}\, d\log(t_a - t_b) \quad \text{if } T_i = \sum t_a \mathrm{Id}_a \right)$$



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E.g. $T_0 = 0$ JMMS $\Leftrightarrow$ Schlesinger equations

$$\begin{tikzcd} W_0 \arrow[loop left, "T_0"] \arrow[rr, bend left, "P"] \arrow[rr, no head] & & W_\infty \arrow[loop right, "T_\infty"] \arrow[ll, bend left, "Q"] \end{tikzcd}$$

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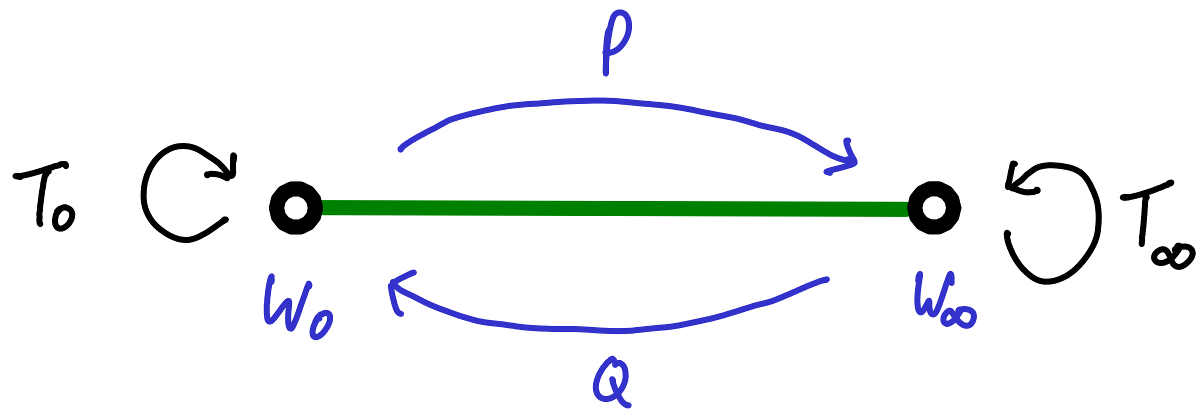
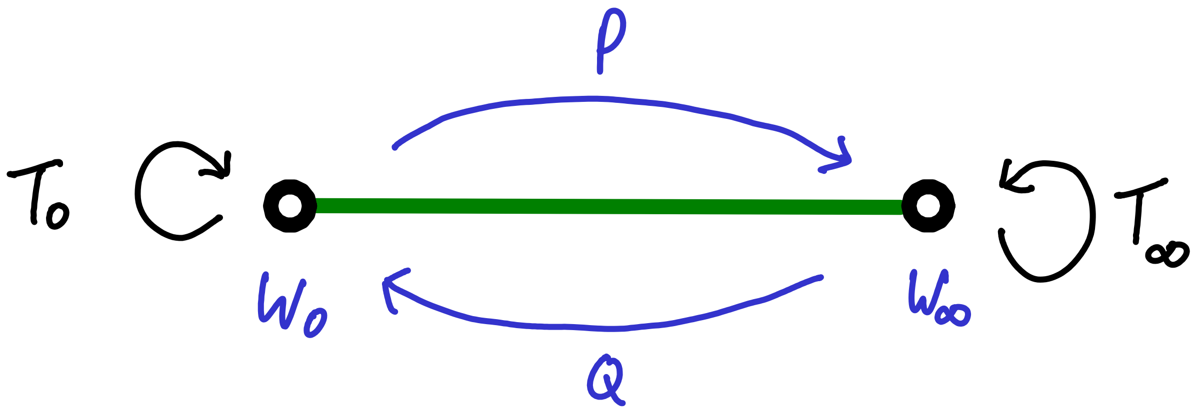
Thm (Harnad '94)

The permutation $(W_0, W_\infty, P, Q, T_0, T_\infty) \longmapsto (W_\infty, W_0, Q, -P, -T_\infty, T_0)$ preserves the JMMS equations



Harnad's duality $(W_0, W_\infty, P, Q, T_0, T_\infty) \longmapsto (W_\infty, W_0, Q, -P, -T_\infty, T_0)$ basically flips over the graph.



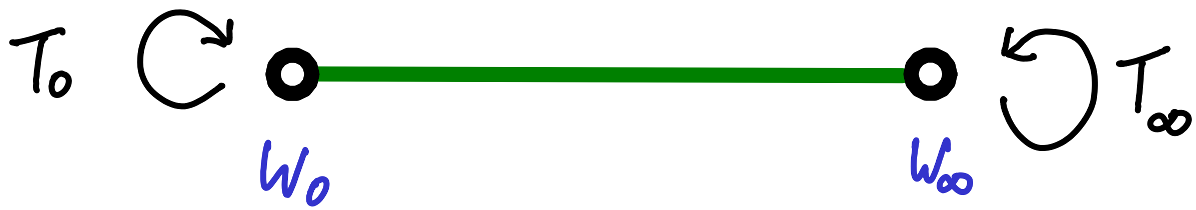
Harnad's duality $(W_0, W_\infty, P, Q, T_0, T_\infty) \longmapsto (W_\infty, W_0, Q, -P, -T_\infty, T_0)$
basically flips over the graph.

JMMS system controls isomonodromic deformations of

$$\left(T_0 + Q\,(z - T_\infty)^{-1} P\right) dz \qquad \text{on} \qquad W_0 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

so $W_0 \leftrightarrow W_\infty$ changes rank of the vector bundle

Splaying / additive fission



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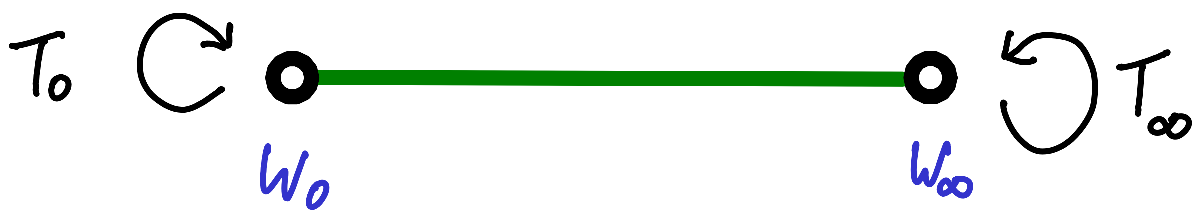

W_0, W_∞ decompose into eigenspaces of T_0, T_∞ :

$$W_j = \bigoplus_{i \in I_j} V_i$$

(I_0, I_∞ label eigenspaces)

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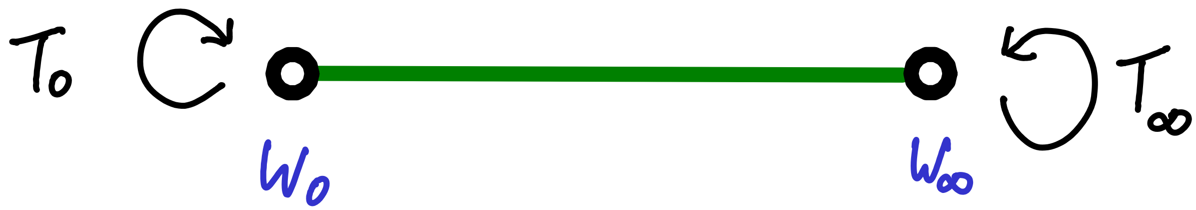

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$$T_j = \sum_{i \in I_j} t_i \, \mathrm{Id}_i \qquad \begin{cases} t_i \in \mathbb{C} & \text{eigenvalues/times} \\ \mathrm{Id}_i = \mathrm{Id}_{V_i} \in \mathrm{End}(W_j) \end{cases}$$

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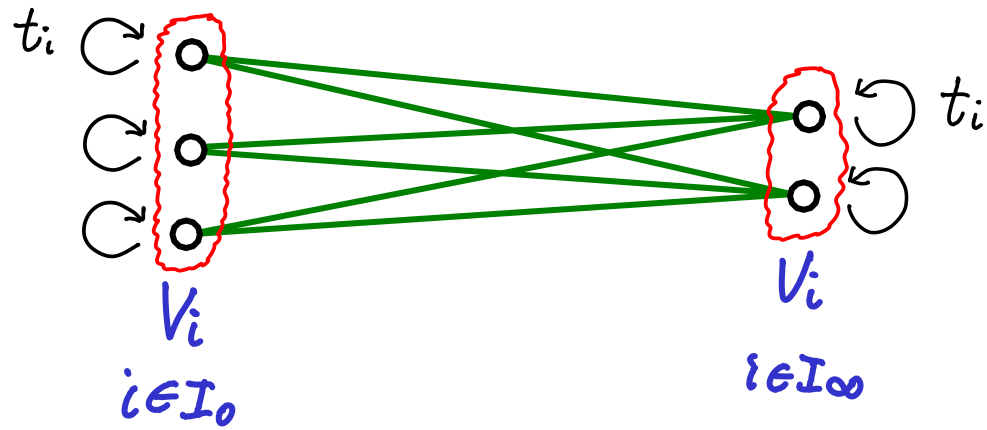

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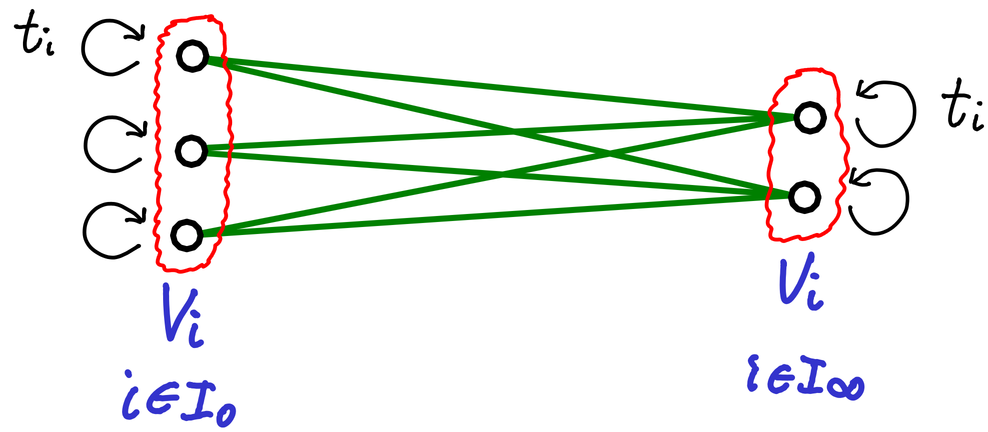

Dependent variables P, Q decompose:

nodes
$I = I_0 \sqcup I_\infty$

$$(P, Q) \iff p_{ij} : V_j \to V_i \quad \forall i, j \in I \text{ s.t. } \exists \text{ edge } i - j$$

$$\mathrm{Hom}(W_0, W_\infty) \oplus \mathrm{Hom}(W_\infty, W_0) \ni (P, Q)$$

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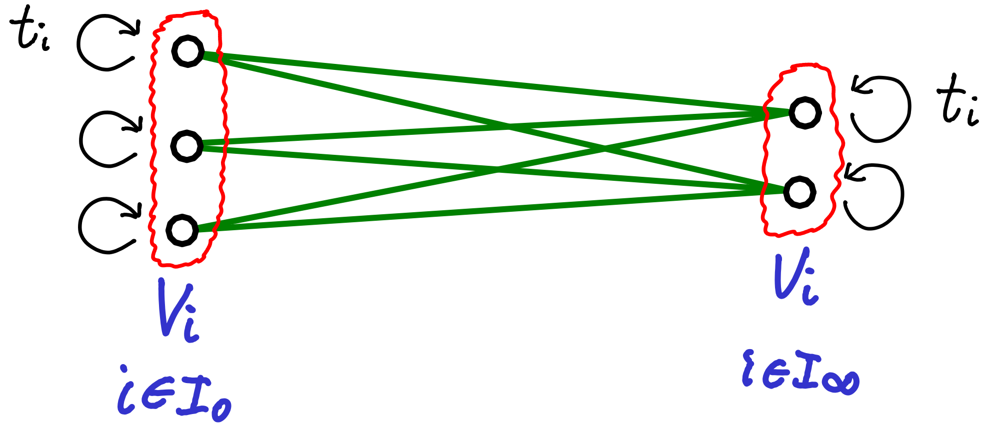

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"Representation of the graph on $V = \bigoplus_{i \in I} V_i$"

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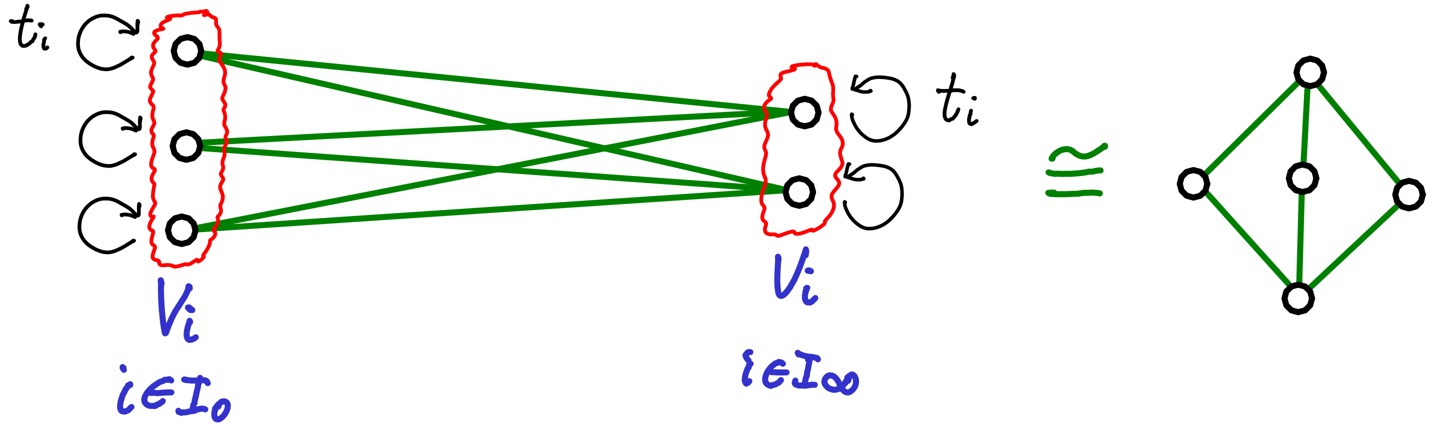
All complete bipartite graphs arise for the JMMS equations:

$\mathcal{G}(3,2) \cong$



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$$\cap\!\!\!\!\mid$$

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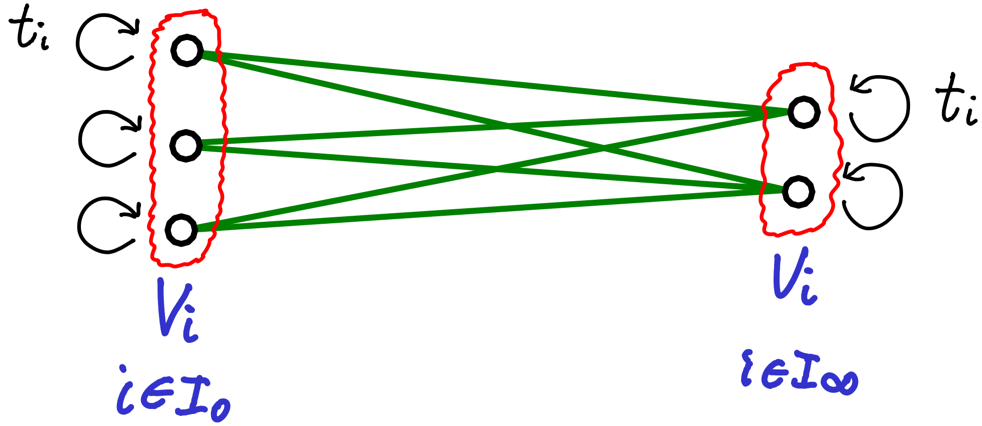

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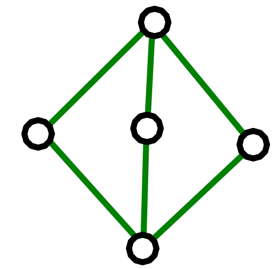

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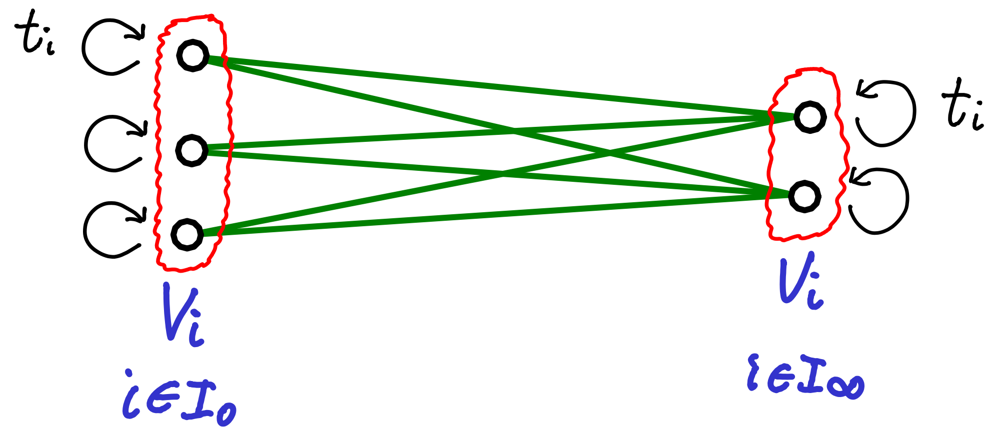

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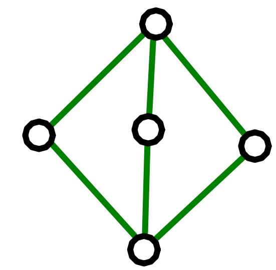
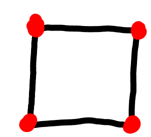


Observe: 1 (JMMS '80) If $|I_0| = |I_\infty| = \dim W_0 = \dim W_\infty = 2$

then JMMS equations $\Longleftrightarrow$ Painlevé V

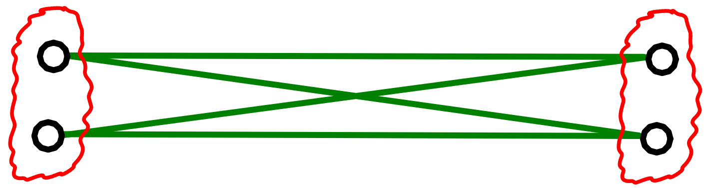
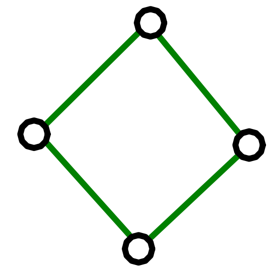
2 (Okamoto '85) Painlevé V has $A_3^{(1)}$ symmetry



3 $\mathcal{G}(2,2)$ is a square

Splaying / additive fission

All complete bipartite graphs arise for the JMMS equations:

$\mathcal{G}(2,2) \cong$  $\cong$ 

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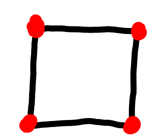
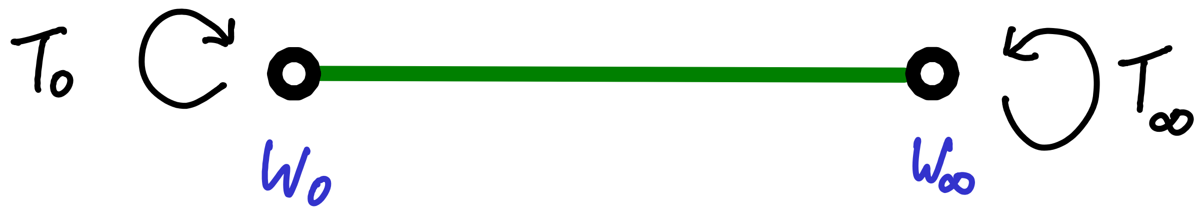
② (Okamoto '85) Painlevé V has $A_3^{(1)}$ symmetry 

③ $\mathcal{G}(2,2)$ is a square

Harnad duality (+ Schles. trsfms) $\Rightarrow$ Okamoto syms

Generalisation:

Replace initial graph



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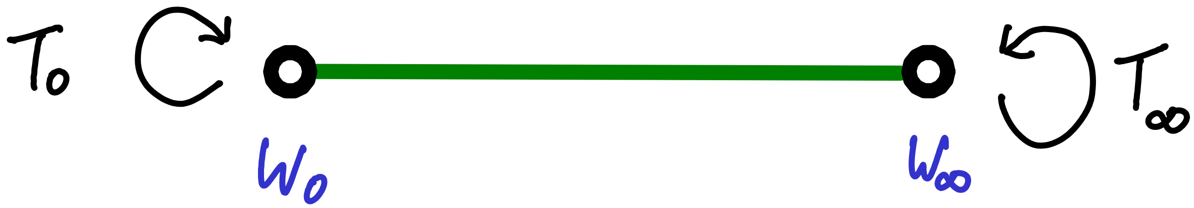
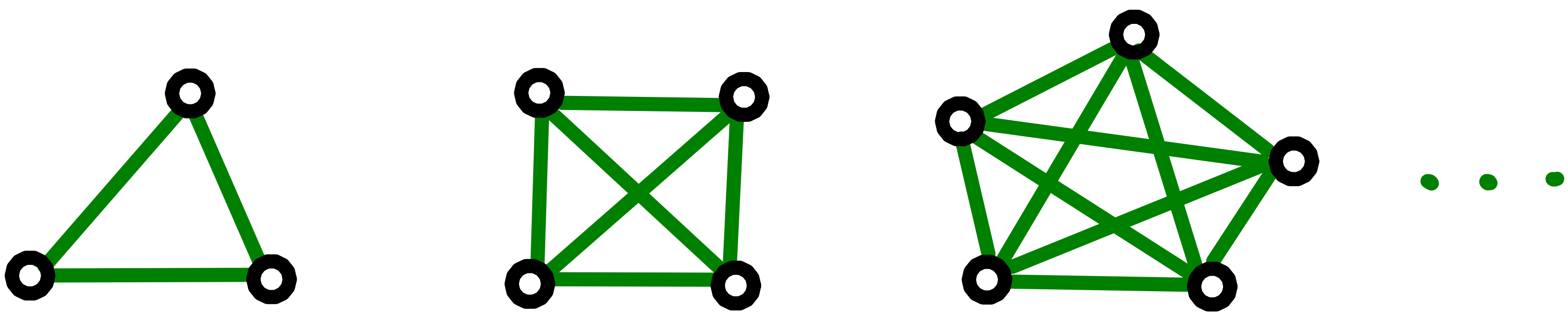


by an arbitrary complete graph:



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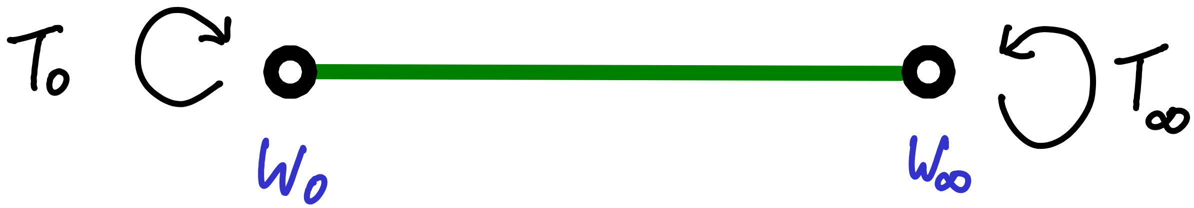

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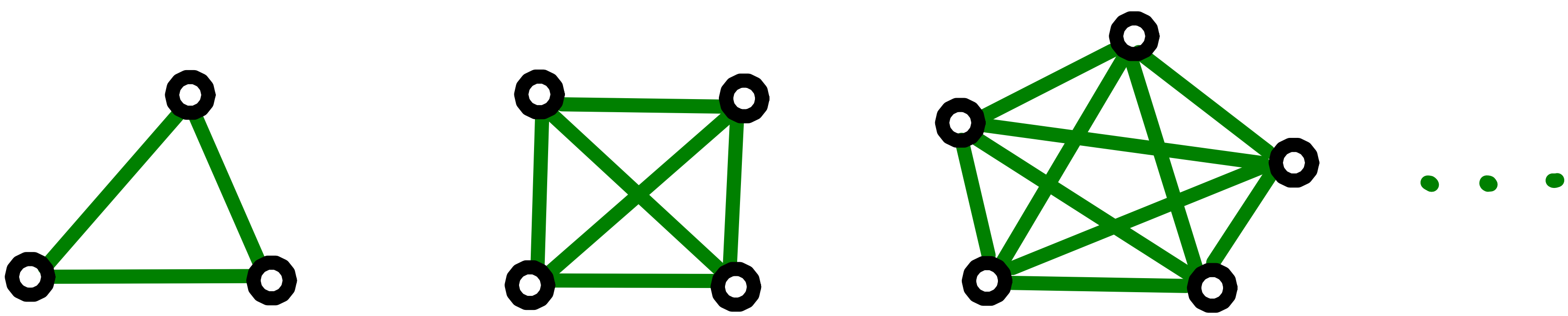

Label nodes by points $J = \{a_j\} \hookrightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$

Put vector spaces W_j at nodes ($\forall j \in J$), & "times" $T_j \in \mathrm{End}(W_j)$ (diagonalisable)

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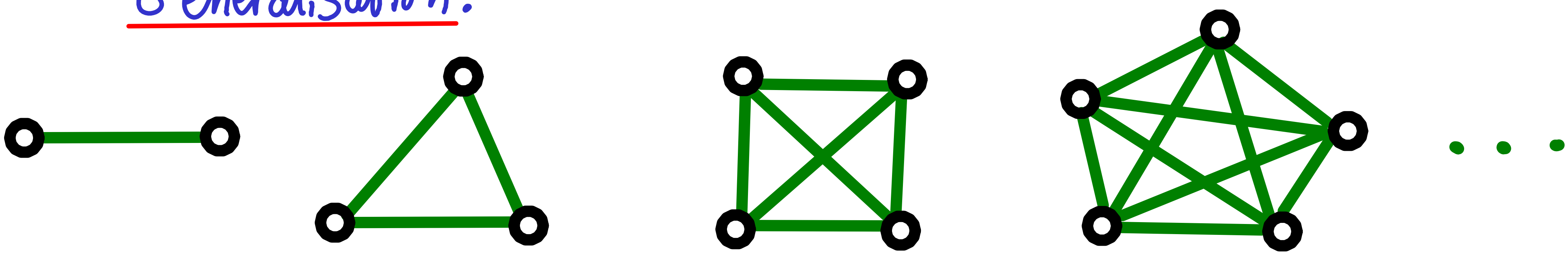

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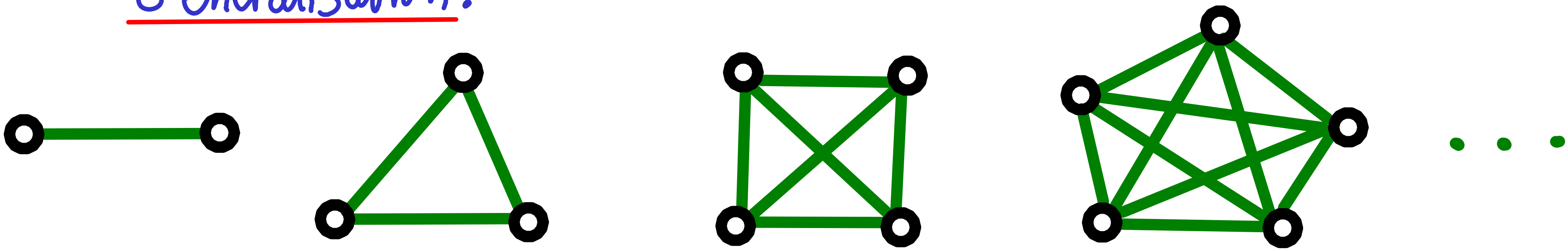

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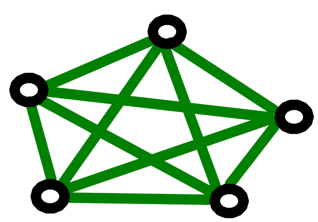
Phase space $\quad \mathcal{M} = \{(P,Q)\} = T^*\mathrm{Hom}(W_0, W_\infty) = \mathrm{Rep}\Big($  $, W\Big)$

$W = W_0 \oplus W_\infty$

$\mathcal{M} = \mathrm{Rep}\Big($  $, W\Big)$, $\quad W = \bigoplus_{j \in J} W_j$

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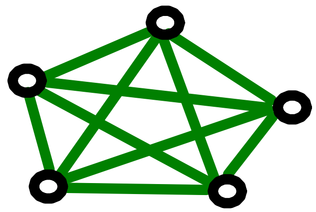
$\mathcal{M} = \mathrm{Rep}\big($  $, W\big)$, $\quad W = \bigoplus_{j \in J} W_j$

Point of $\mathcal{M}$ consists of maps $B_{ij} : W_j \to W_i \quad \forall i \neq j \in J$

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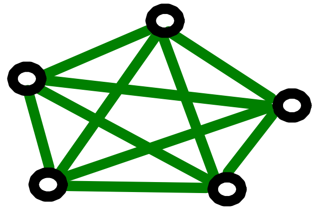
$\mathcal{M} = \mathrm{Rep}\Big($  $, W\Big)$, $W = \bigoplus_{j \in J} W_j$

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<u>Thm</u> • Have (integrable) isomonodromy system

for $\Gamma = \{B_{ij}\}$ w.r.t $\underset{\sim}{T} = \{T_j\}$

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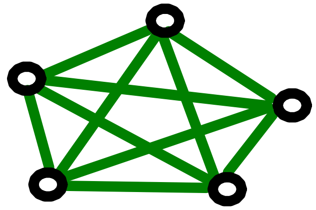
$\mathcal{M} = \mathrm{Rep}\Big($  $, W\Big)$, $W = \bigoplus_{j \in J} W_j$

Point of $\mathcal{M}$ consists of maps $B_{ij} : W_j \to W_i$ $\forall\, i \neq j \in J$

<u>Thm</u> • Have (integrable) isomonodromy system for $\Gamma = \{B_{ij}\}$ w.r.t $\underset{\sim}{T} = \{T_j\}$

• Governs isomonodromic deformations of linear differential systems on $\Big(\bigoplus_{j \neq \infty} W_j\Big) \times \mathbb{P}^1 \to \mathbb{P}^1$

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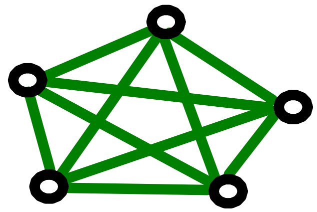
$\mathcal{M} = \mathrm{Rep}\Big($  $, W\Big)$, $\quad W = \bigoplus_{j \in J} W_j$

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Thm
- Have (integrable) isomonodromy system for $\Gamma = \{B_{ij}\}$ w.r.t. $\underset{\sim}{T} = \{T_j\}$
- Governs isomonodromic deformations of linear differential systems on $\Big(\bigoplus_{j \neq \infty} W_j\Big) \times \mathbb{P}^1 \to \mathbb{P}^1$
- Can act by Möbius transforms on $J \subset \mathbb{P}^1$ to get equiv. system

$J = \{a_j\} \hookrightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, times $T_j \in \mathrm{End}(W_j)$ (diagonalisable)

$$B_{ij} : W_j \to W_i \quad \forall i \neq j \in J$$

Simply-laced isomonodromy system:

$$dB_{ij} = \sum_{k \in J} \widetilde{X_{ik} B_{ki}} B_{ij} + B_{ij} \widetilde{B_{jk} X_{kj}}$$

$$+ dT_i X_{ik} B_{kj} + B_{ik} X_{kj} dT_j - X_{ik} dT_k X_{kj} / \phi_{ij}$$

$$+ \text{linear terms}$$

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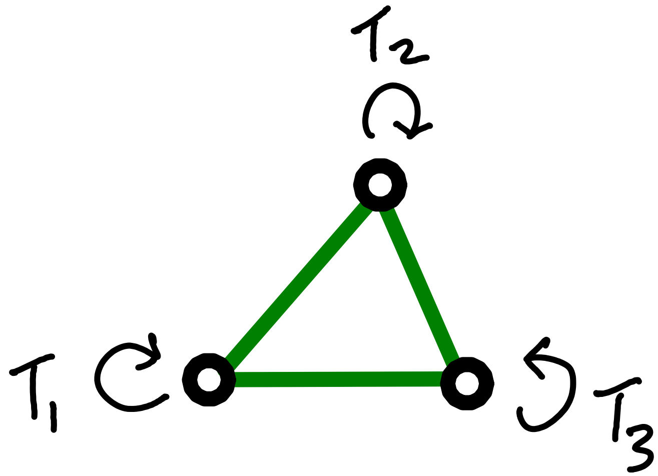
$$+ \text{ linear terms}$$

where

$$\phi_{ij} = \begin{cases} (a_i - a_j)^{-1} & \text{if } i,j \neq \infty \\ 1, -1 & j = \infty, i = \infty \text{ resp.} \end{cases}$$

$$X_{ij} = \phi_{ij} B_{ij}, \qquad (B_{ii} = 0)$$

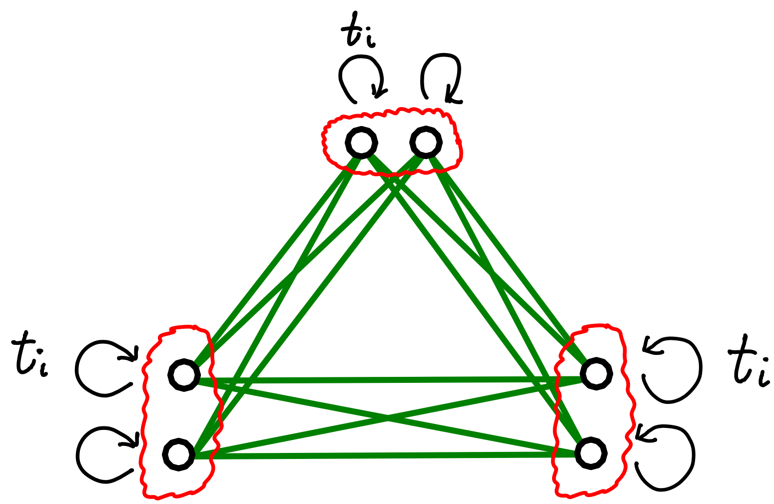
Splay/fission as before:



$$I_j = \mathrm{Eigenspaces}(T_j)$$

$$\Downarrow$$

e.g. $|I_j| = 2 \quad \forall j$:

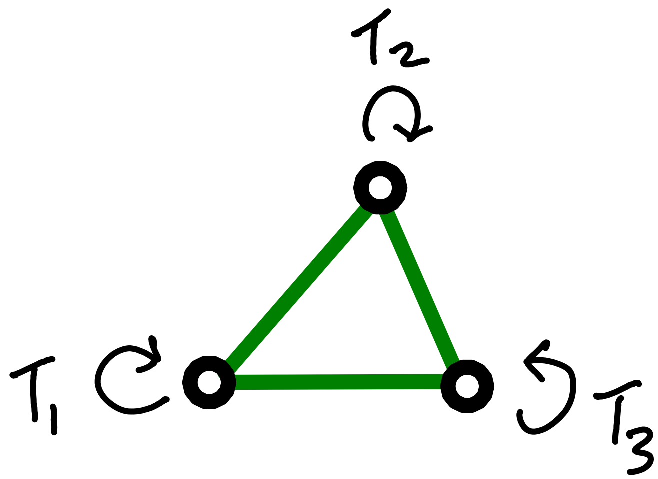


nodes

$$I = \bigsqcup_{j \in J} I_j$$

$$\bigoplus_{i \in I_j} V_i = W_j$$

Splay/fission as before:



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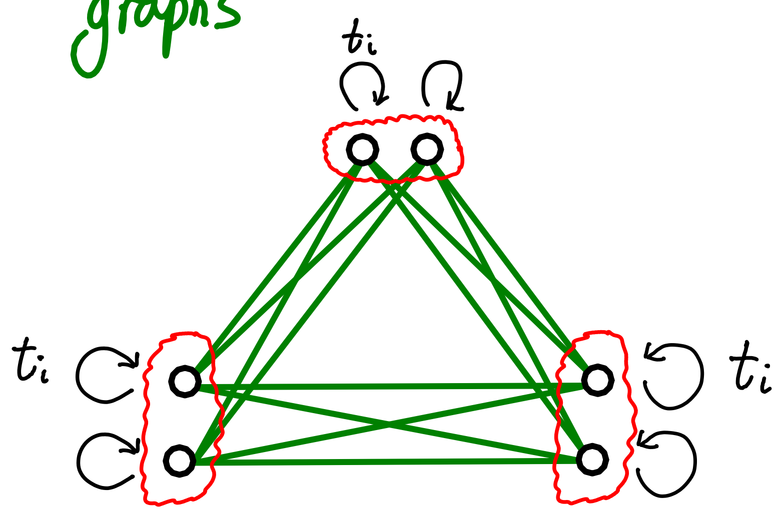
Get all complete k-partite graphs

$k = |J| = \#\text{nodes}$

e.g $|I_j| = 2 \ \forall_j$:

$\mathcal{G}(2,2,2) \cong$



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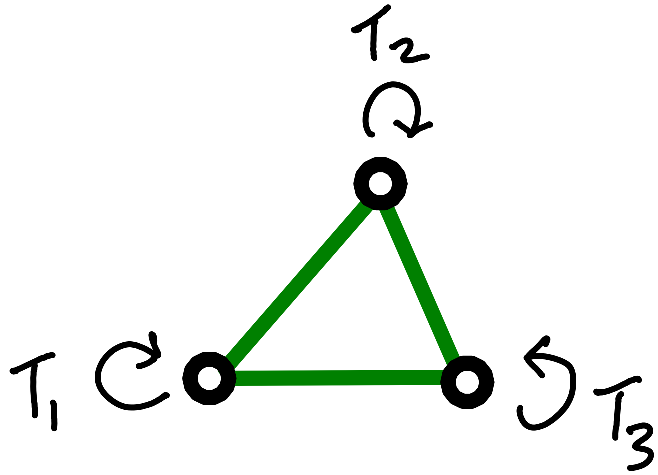

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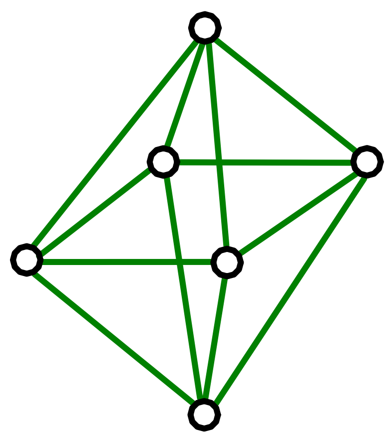
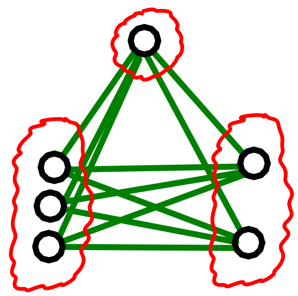
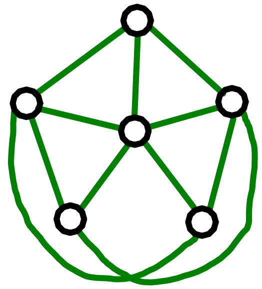


Complete k partite graphs $\Longleftrightarrow$ Integer partitions with k parts



$\Longleftrightarrow$ $1+2+3=6$

Complete k partite graphs $\Longleftrightarrow$ Integer partitions with k parts

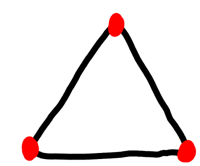


$\Longleftrightarrow$ $1+2+3=6$

E.g. Observe: 1 If $k=|J|=3$, $\dim W_0 = \dim W_0 = \dim W_\infty = 1$

then s. laced IM system $\Leftrightarrow$ Painlevé IV

2 (Okamoto '85) Painlevé IV has $A_2^{(1)}$ symmetry

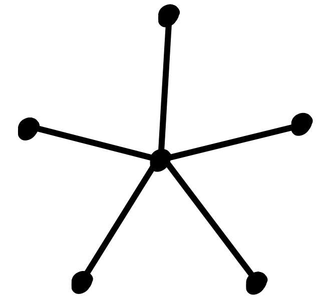


3 $\mathcal{G}(1,1,1)$ is a triangle

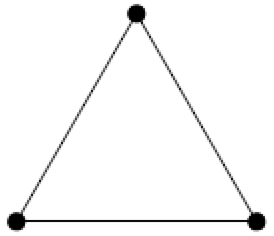
Möbius$(SL_2(\mathbb{C}))$ symmetries (+ Schles. trfms) $\Rightarrow$ Okamoto syms

Graphs from partitions of $N \leq 6$

(omitting totally disconnected graphs $\mathcal{G}(n)$, and stars $\mathcal{G}(n,1)$)



5,1



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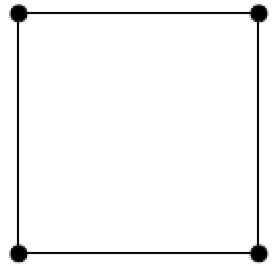

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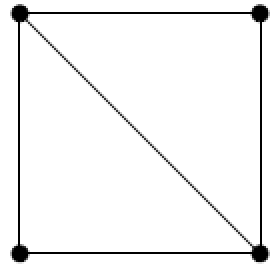
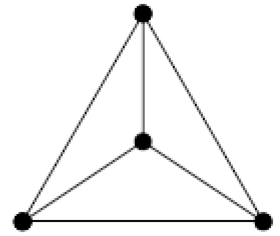


211



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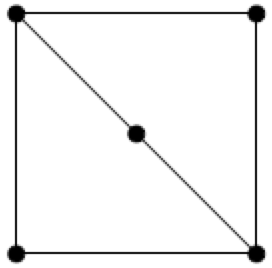

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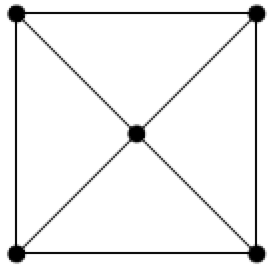

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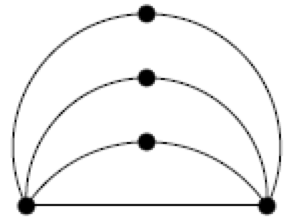

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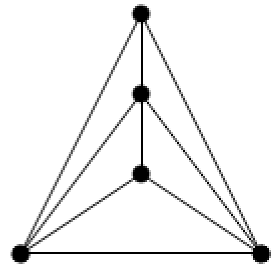

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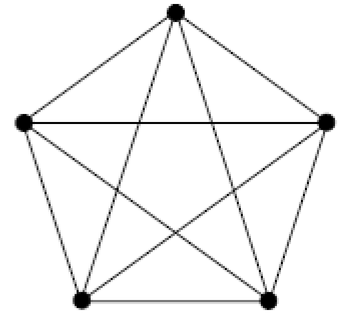

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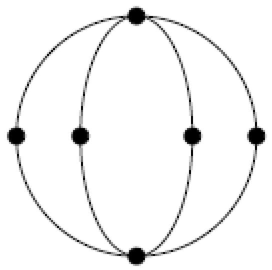

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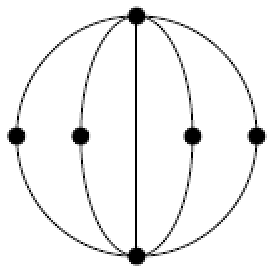

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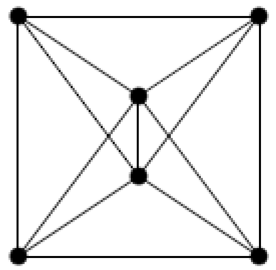

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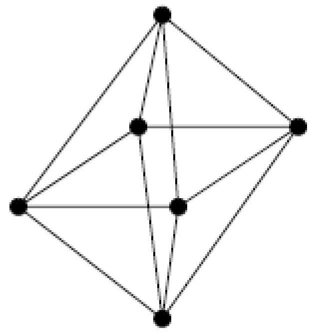

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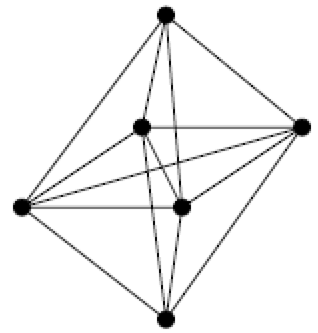

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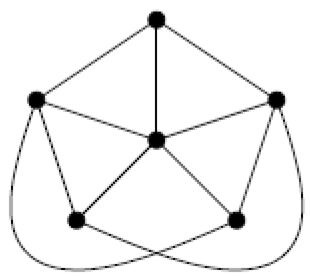

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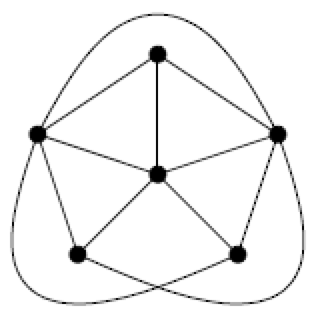

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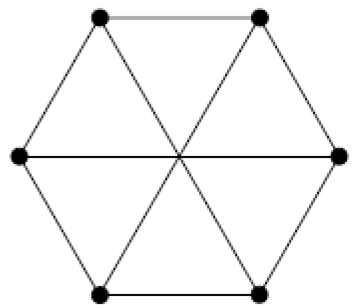

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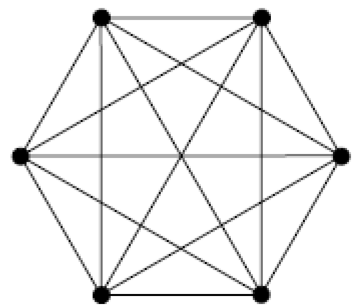

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Ref.s $\left\{\begin{array}{l}\text{arXiv: 0806.1050 Irregular con}^{\text{n}}\text{s + KM root systems} \\ \text{Pub.Math. IHES 2012 Simply-laced isomonodromy systems}\end{array}\right.$

Further steps

- Main idea — presentations of modules for Weyl algebra $\mathcal{A}_1$
- Reductions; reduced phase space $\mathcal{M}^* \cong$ Nakajima quiver var.
  $\cong$ moduli of connections
  $\cong$ moduli of $\mathcal{A}_1$-module pres$^{\text{n}}$s
- Weyl group action via reordering eigenvalues of residues
- Hamiltonians and τ-functions
- Examples: Higher Painlevé systems

Main idea

Let $\mathcal{A}_1 = \mathbb{C}\langle z, \partial \rangle$, $\partial = d/dz$

Suppose α, β, γ $n \times n$ matrices $/ \mathbb{C}$

Let $M = \alpha \partial + \beta z - \gamma$

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Suppose $\boxed{\alpha, \beta \text{ commuting semisimple}, \ \ker(\alpha) \cap \ker(\beta) = 0}$

Consider $\mathcal{A}_1$ modules $\mathcal{N}$ of form

$$\mathcal{A}_1^n \xrightarrow{(\cdot M)} \mathcal{A}_1^n \longrightarrow \mathcal{N} \longrightarrow 0$$

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Lemma This class of modules is stable under the $SL_2(\mathbb{C})$ (symplectic) symmetries of $\mathcal{A}_1$

Everything follows from this (JMMS case ~ $\ker(\alpha) \oplus \ker(\beta) = V = \mathbb{C}^n$)

$V = \mathbb{C}^n$ decomposes into joint eigenspaces of α, β

Eigenvalues $\alpha_i, \beta_i \implies$ point $a_i = -\beta_i/\alpha_i \in \mathbb{P}^1 = \mathbb{C} \cup \infty$

$$V = \bigoplus_{a \in \mathbb{P}^1} W_a = \bigoplus_{j \in J} W_j \qquad (J = \{a_i \text{ occurring}\})$$

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$$\gamma = \underbrace{(T_i)}_{\mathrm{End}(W_i)} + \underbrace{(B_{ij})}_{\mathrm{Hom}(W_j, W_i)\ i \neq j} \in \mathrm{End}\left(\bigoplus W_j\right)$$

(assume T_i semisimple)

$$= \begin{pmatrix} C & 0 \\ 0 & T \end{pmatrix} + \begin{pmatrix} 0 & P \\ Q & B \end{pmatrix} \in \mathrm{End}(W_\infty \oplus U_\infty)$$

$$U_\infty = \bigoplus_{j \neq \infty} W_j \quad , \quad C = T_\infty$$

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Solutions of $\mathcal{N}$ $\Longleftrightarrow$ solutions of $\mathrm{End}(U_\infty)$ system:

$$\partial v = \left(Az + B + T + Q(z-c)^{-1}P\right) v$$

$$A = \sum_{j \neq \infty} a_j \, \mathrm{Id}_{W_j}$$

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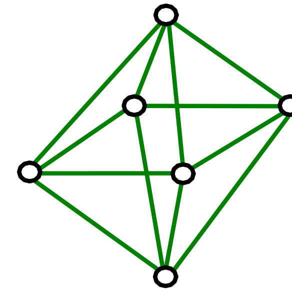
- $SL_2(\mathbb{C})$ action acts by Möbius trfmns on $\{a_j\} \subset \mathbb{P}^1$
- $U_\infty = \bigoplus_{a_j \neq \infty} W_j$ (so rank changes $\Rightarrow$ $k+1$ different values in general) $k = |J|$

Reduction

$$\mathbb{M} = \{ B_{\bullet\bullet} \} \cong \{ \rho_{ij} \} = \operatorname{Rep}(\mathcal{G}, V) \quad \Big(\mathcal{G} = $$  $$\text{say} \Big)$$

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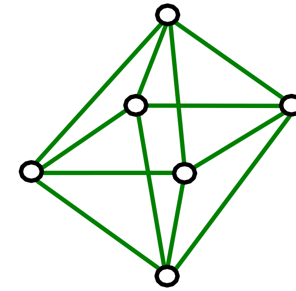

$$\left. \text{say} \right)$$

Have action of reductive group $\hat{H} = \prod_{i \in I} GL(V_i)$ preserving SLIM equations

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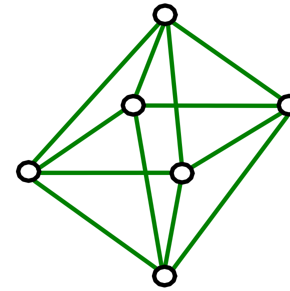
($\mathcal{G} =$  say)

Have action of reductive group $\hat{H} = \prod_{i \in I} GL(V_i)$ preserving SLIM equations

Choose (co)adjoint orbit $\check{\mathcal{O}} \subset \mathrm{Lie}(\hat{H}) = \prod \mathfrak{gl}(V_i)$

(i.e. $\check{\mathcal{O}}_i \subset \mathfrak{gl}(V_i) \quad \forall\, i \in I$)

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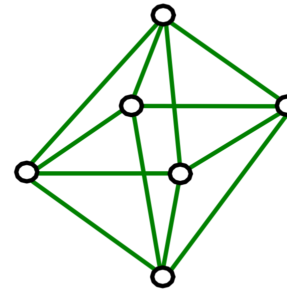
$\mathcal{M} = \{ B_{\cdot\cdot} \} \cong \{ \rho_{ij} \} = \mathrm{Rep}(\mathcal{G}, V)$ $\Big(\mathcal{G} =$  say $\Big)$

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(i.e. $\check{\mathcal{O}}_i \subset \mathfrak{gl}(V_i)$ $\forall\, i \in I$)

Let $\mathcal{M}^* = \mathcal{M} /\!/_{\check{\mathcal{O}}} \hat{H}$ (symplectic quotient)

— Reduced phase space (really look at stable points)

$\rightsquigarrow \mathcal{M}^* \cong$ a Nakajima quiver var.

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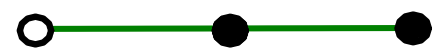
It is known that $\check{\mathcal{O}}_i \subset GL(V_i)$ is a quiver variety

$\check{\mathcal{O}}_i \longleftrightarrow$ V_i

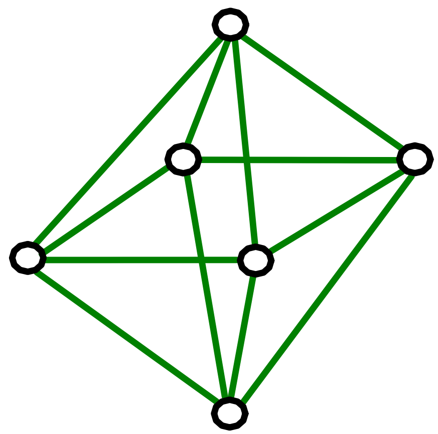


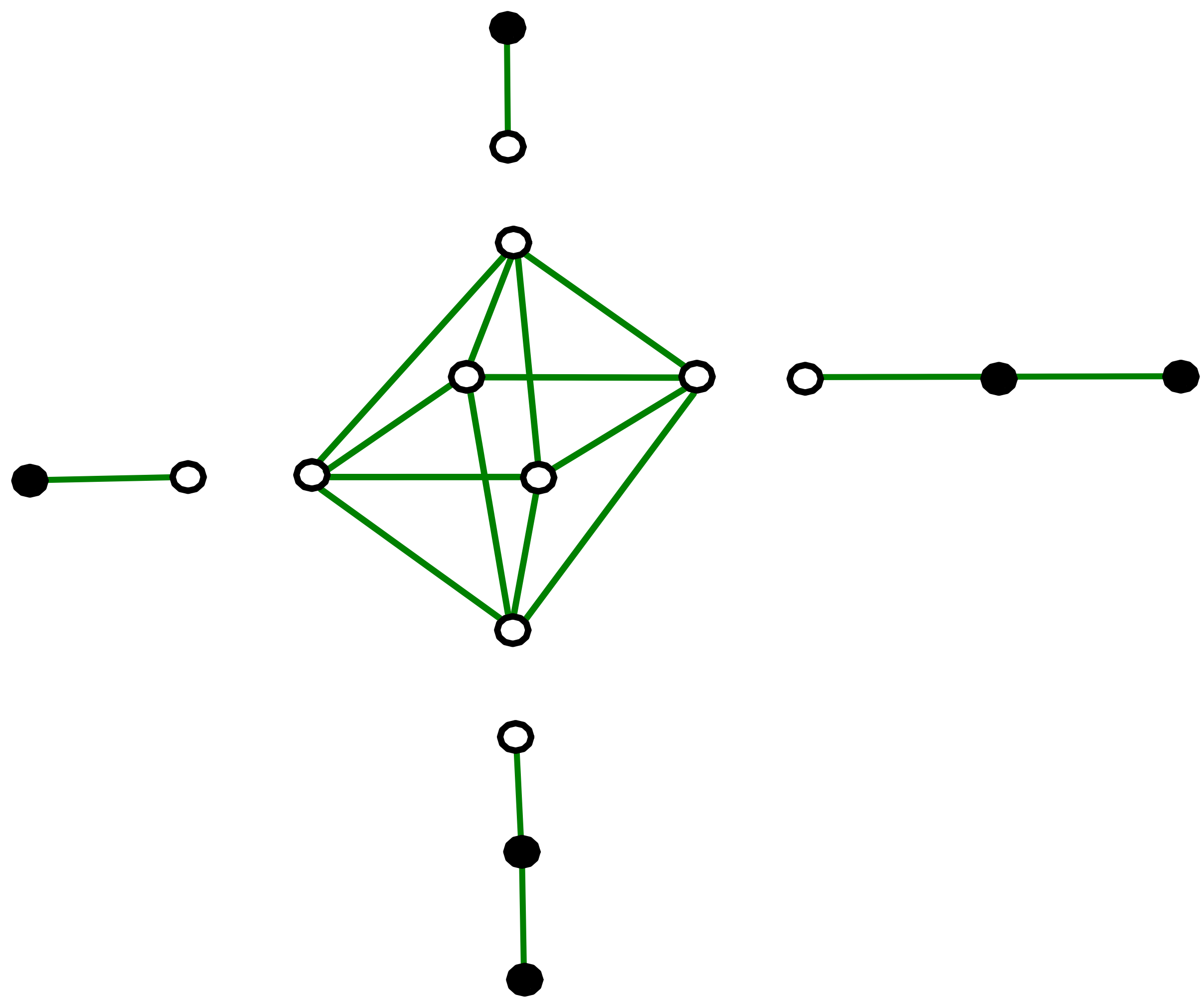
(Kraft-Procesi, ...)

- depends on choice of order of roots of minimal polynomial (of elements of $\check{\mathcal{O}}_i$)

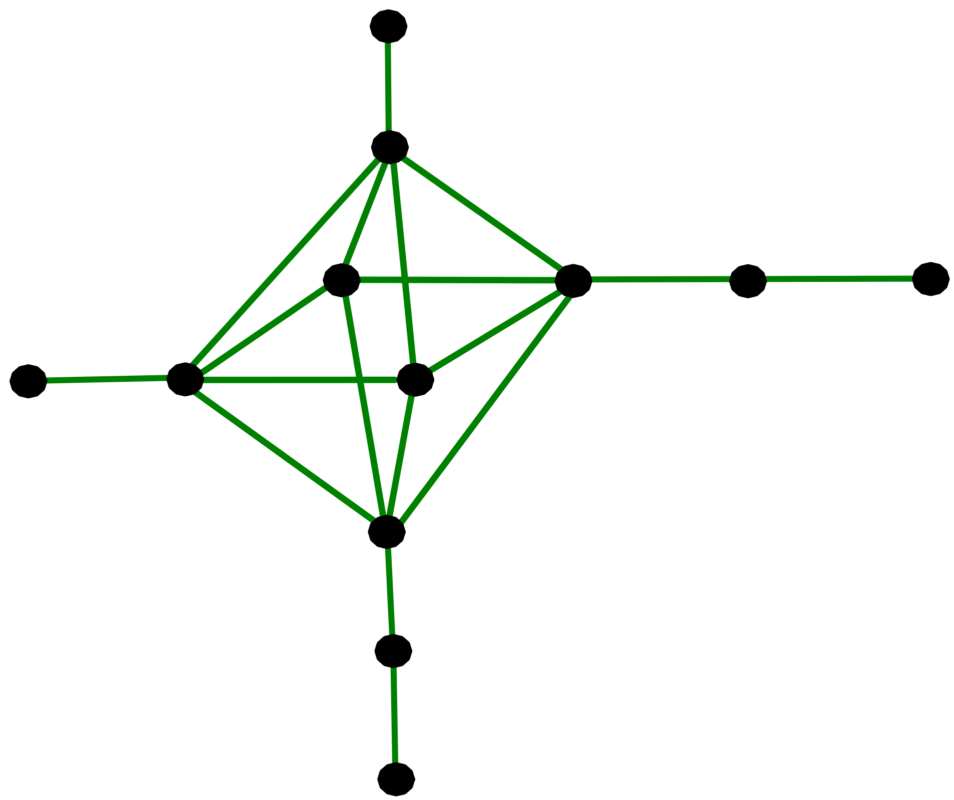
- glue such "legs" on to $\mathcal{G}$

$\mathcal{G} =$





$\hat{\mathcal{G}} =$



$\mathcal{M}^* \cong \text{QuiverVar.}(\hat{\mathcal{G}})$

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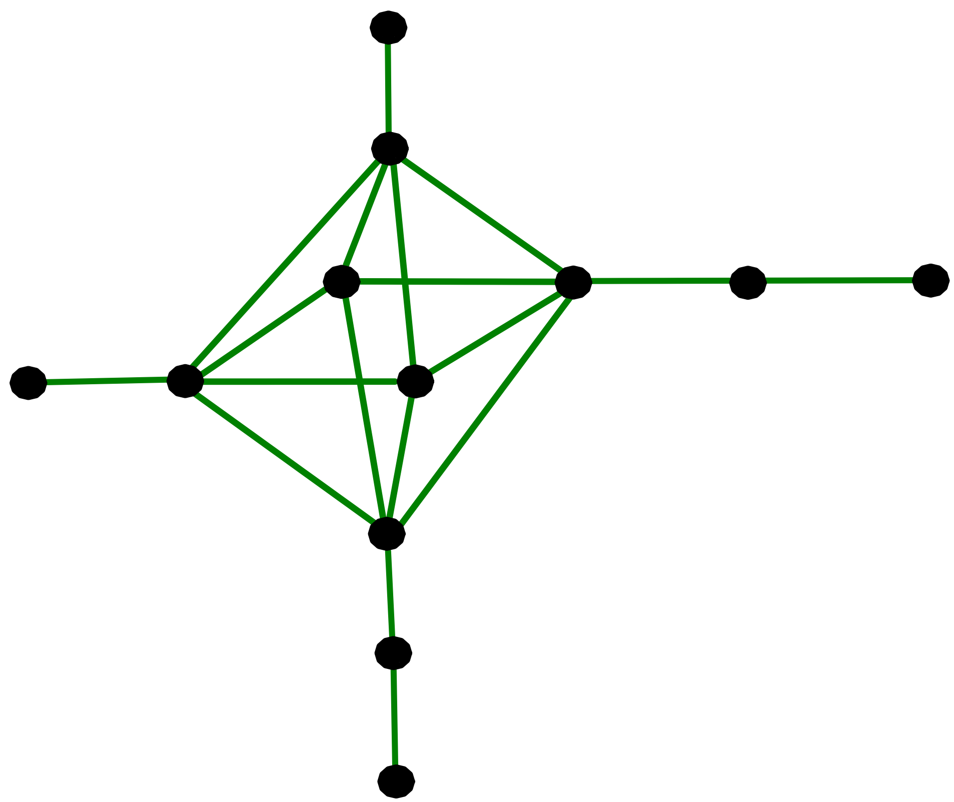
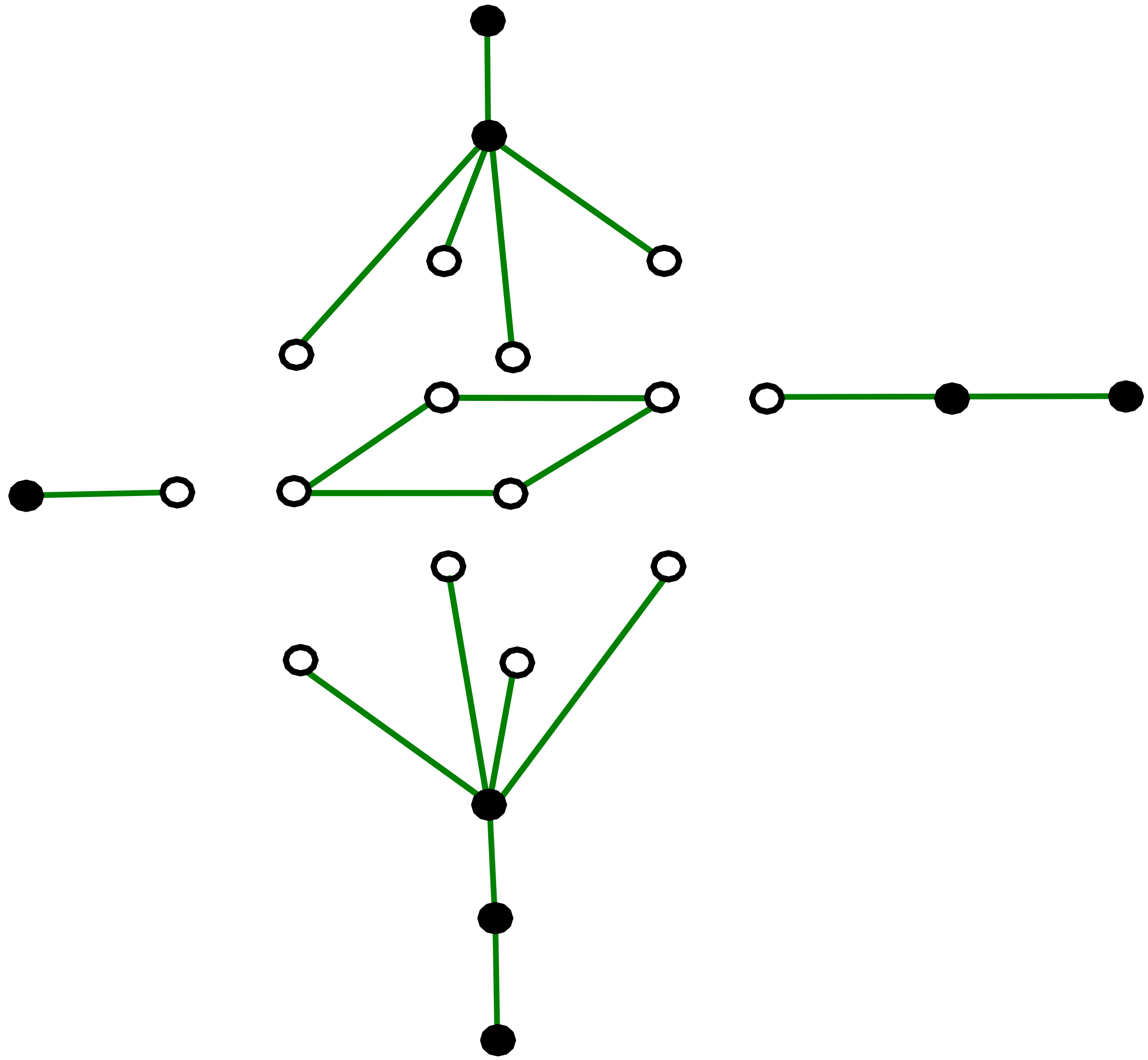
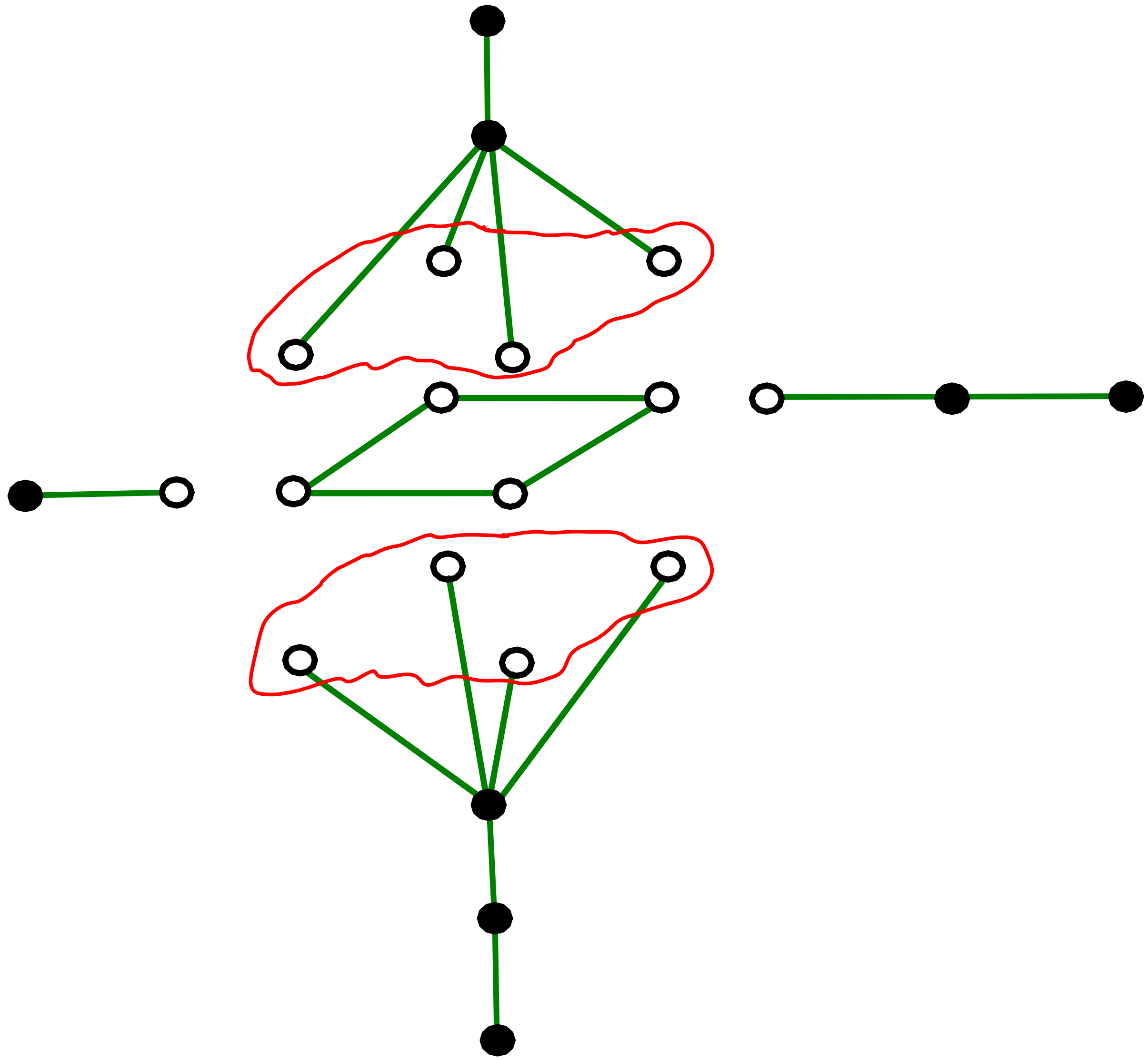


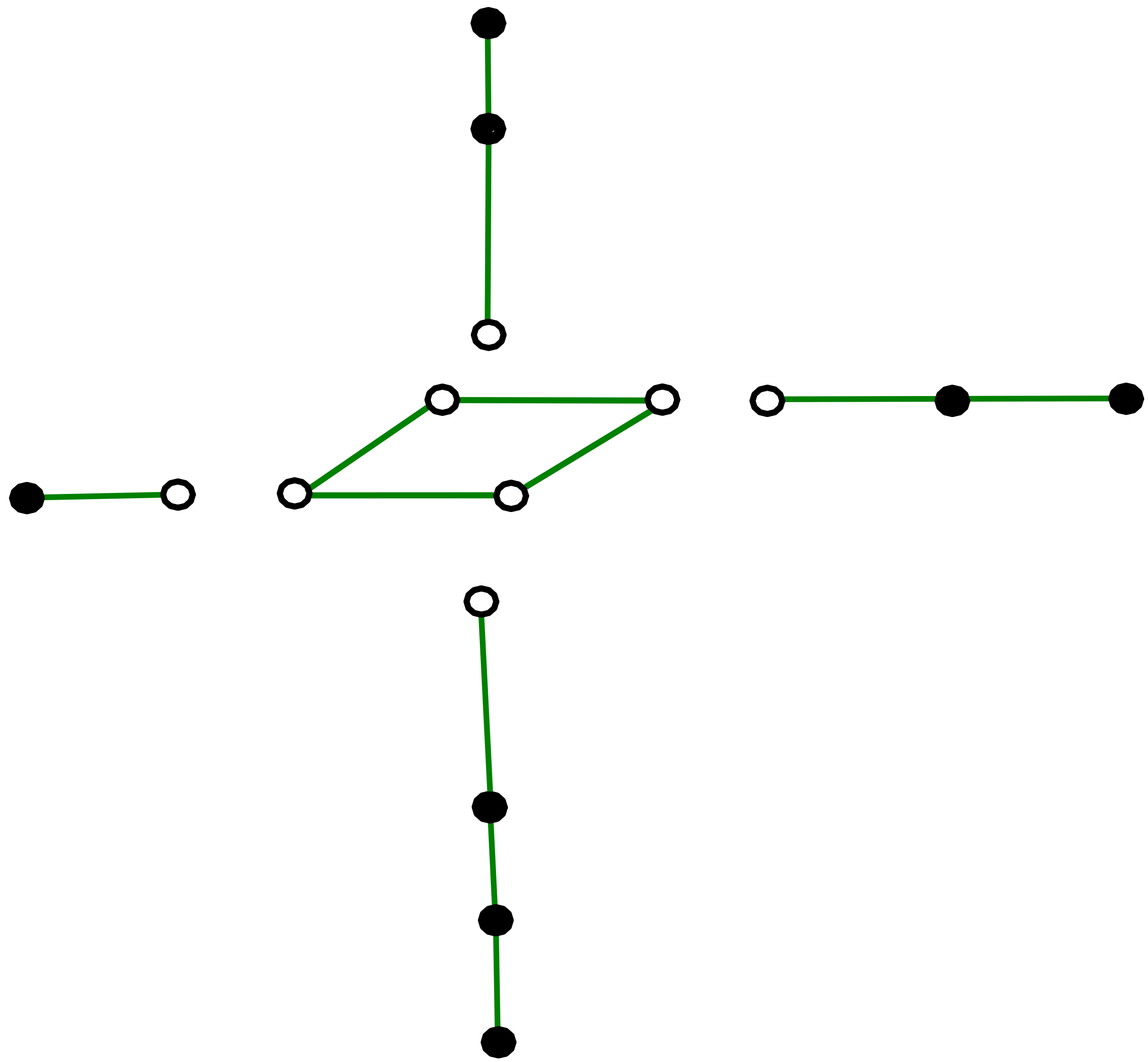
$$\mathcal{M}^* \cong \text{QuiverVar.}(\hat{\mathcal{G}})$$

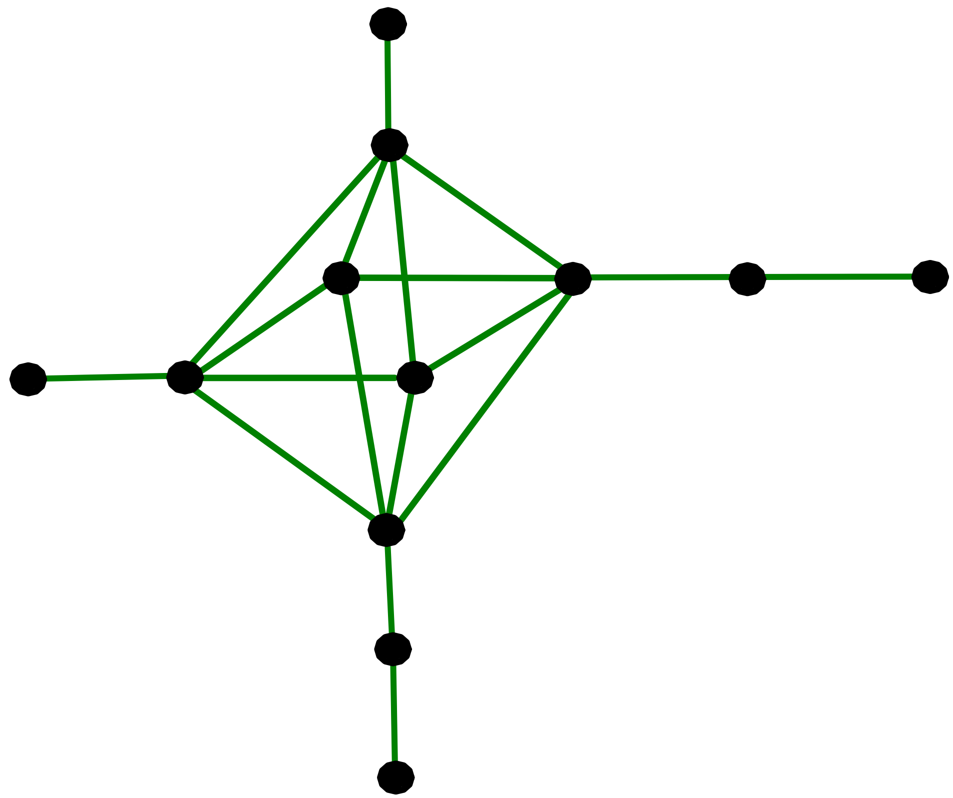
Class of graphs that appear = "Supernova graphs" (complete k-partite + legs)

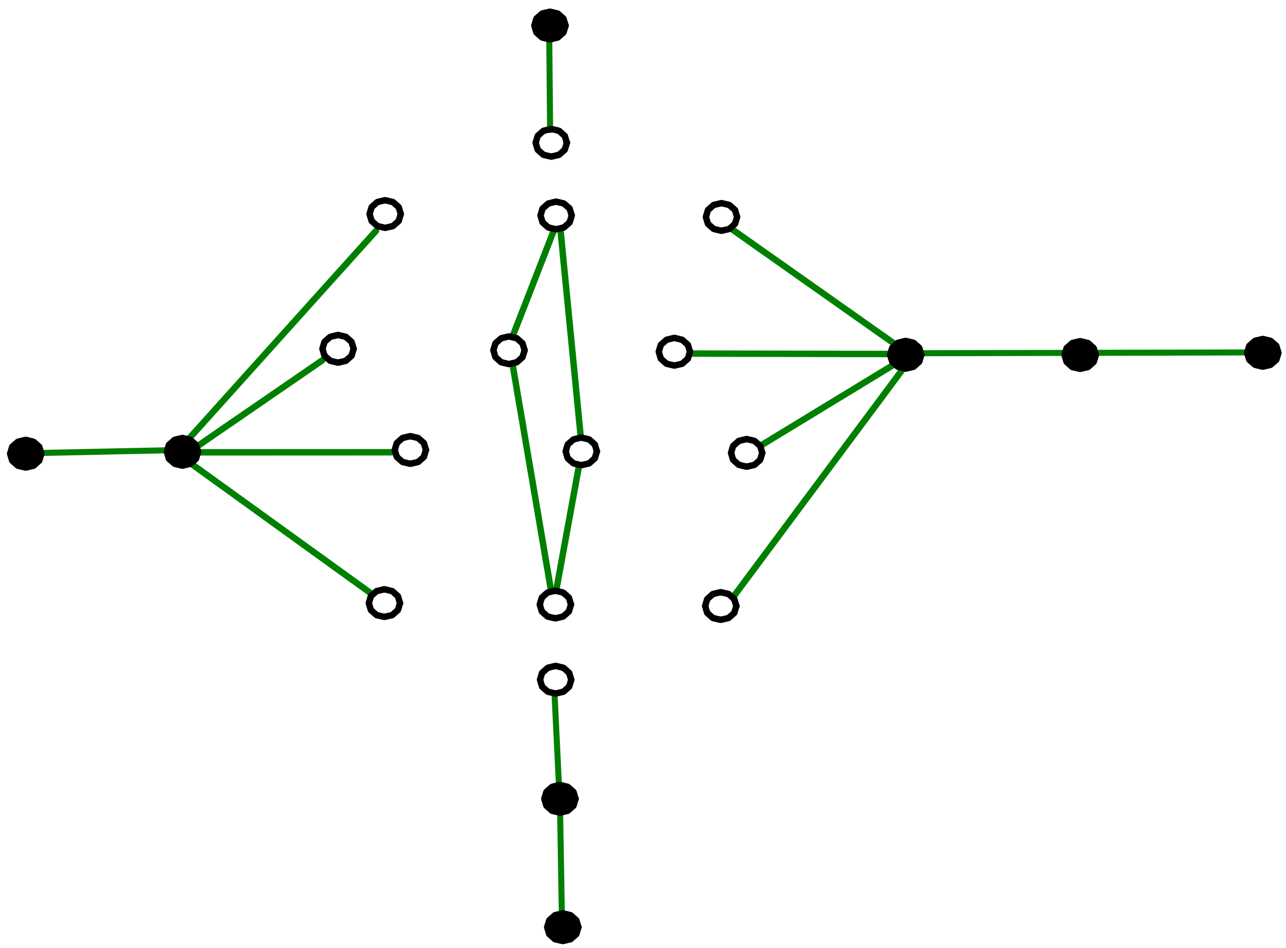
- can attach isomonodromy system to any such graph & its Weyl group gives isomorphisms

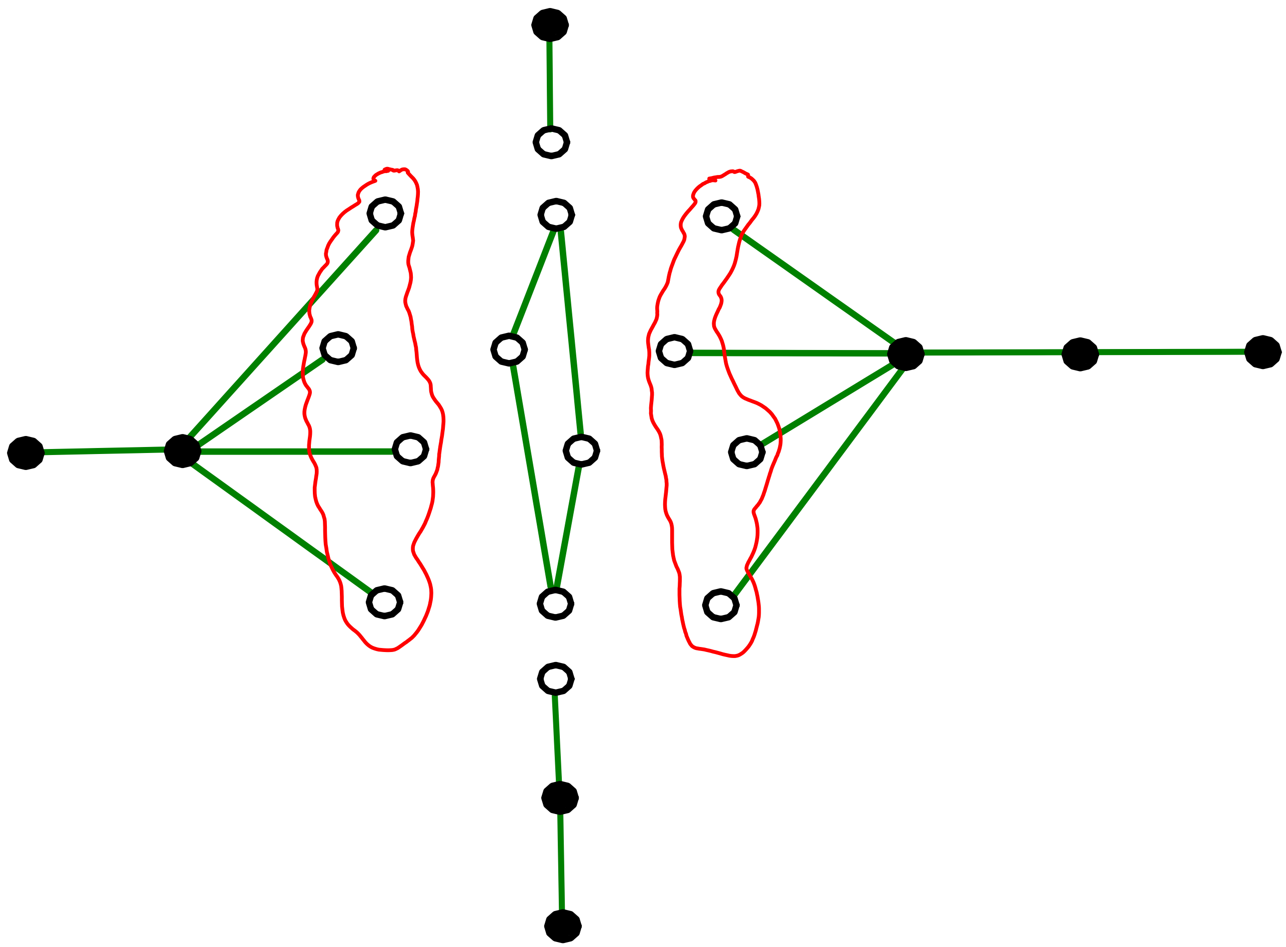


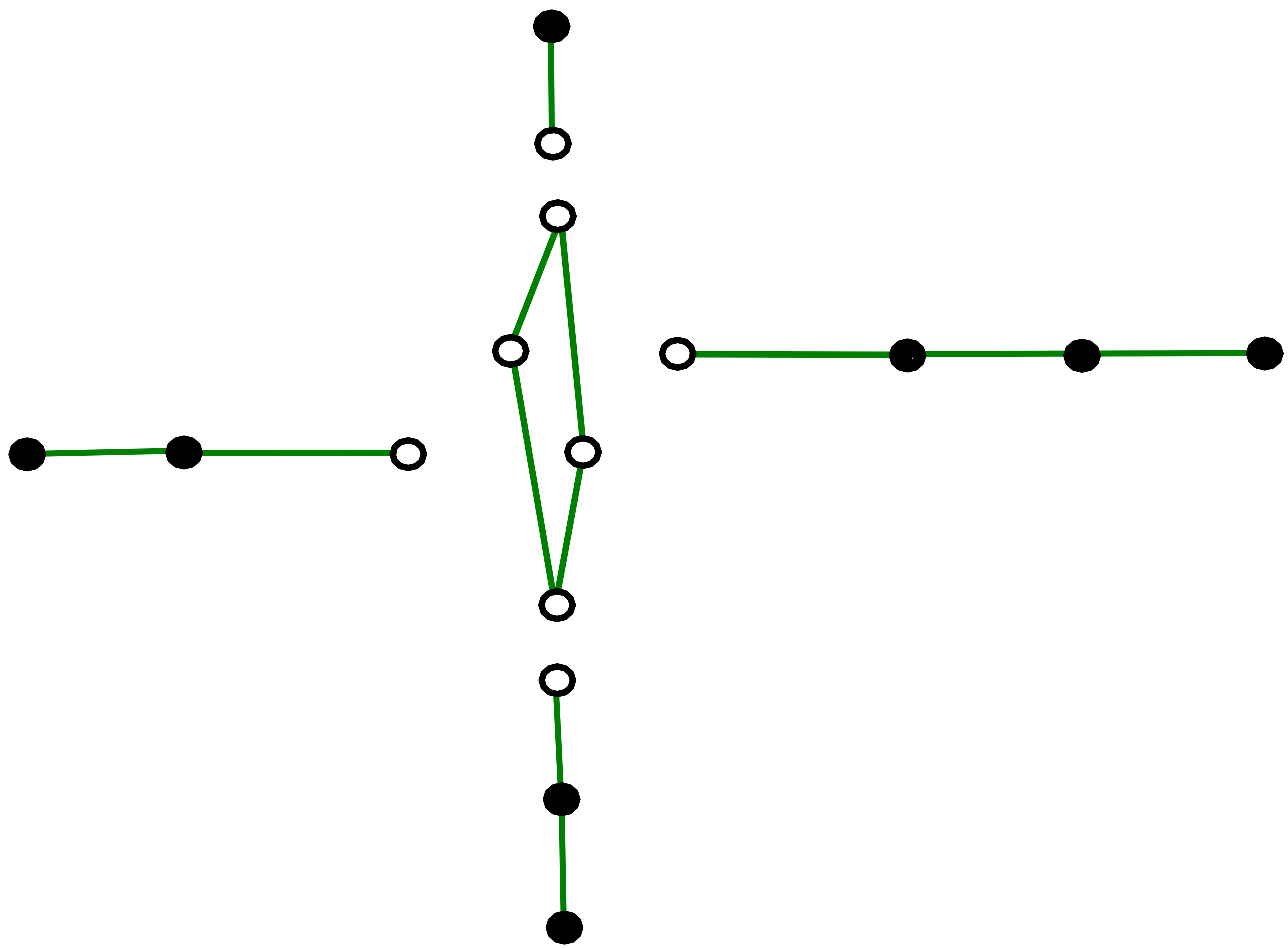


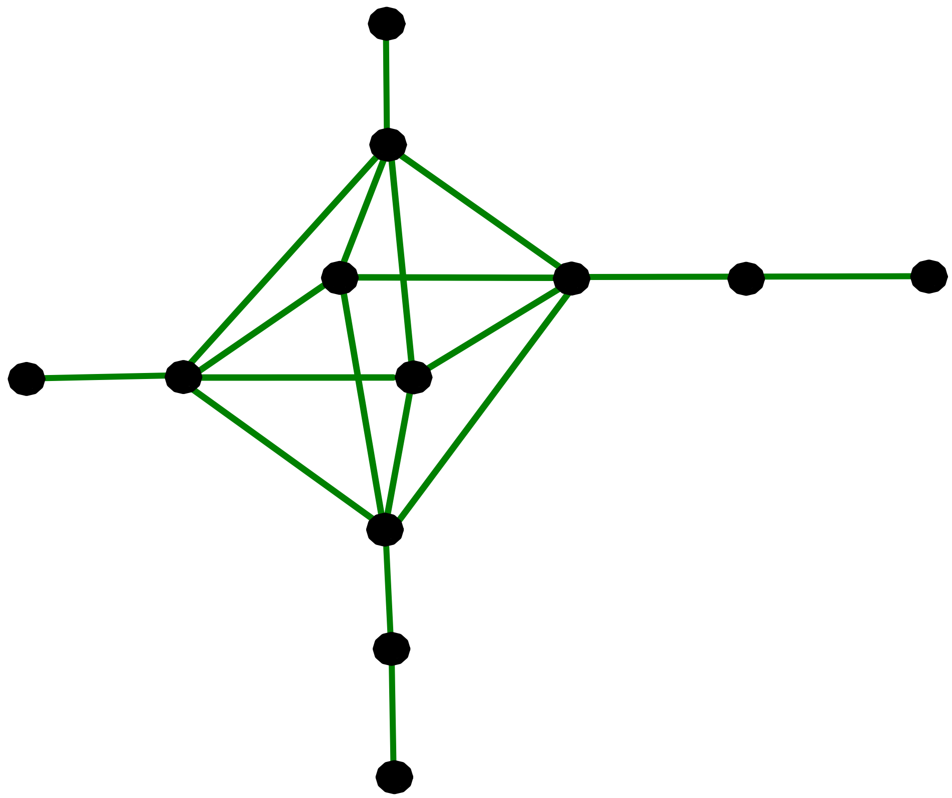












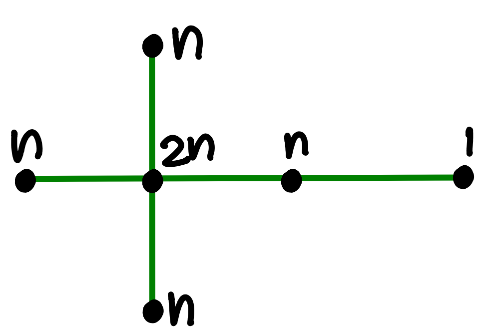
Examples: Higher Painlevé systems

"For any of the (classical, 2nd order) Painlevé equations $X = \mathrm{I}, \mathrm{II}, \dots, \mathrm{VI}$ there is an isomonodromy system hP_X^n of order $2n$ $\forall n = 1, 2, \dots$"

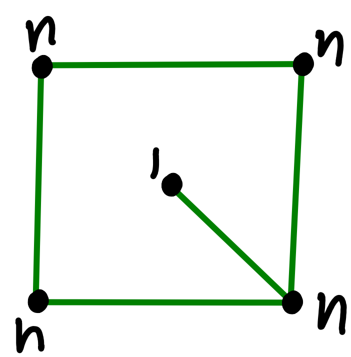
Examples: Higher Painlevé systems

"For any of the (classical, 2nd order) Painlevé equations $X = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VI}$ there is an isomonodromy system hP^n_X of order $2n$ $\forall n = 1, 2, \ldots$"

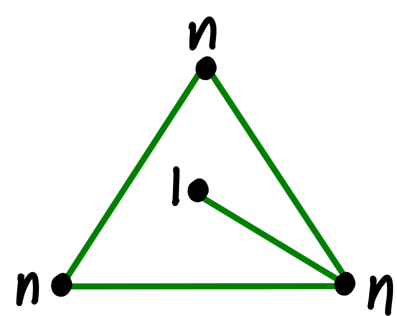
Supernova graph $\hat{\mathcal{G}}$ + vector of dimensions $\Rightarrow$ IMD system



hP^n_{VI}



hP^n_{V}



hP^n_{IV}

complex dimension two, which are related to affine algebras. For example finding spaces of stable connections of complex dimension 4 then (after Theorem 3 below) basically amounts to finding integral vectors of norm -2. (There are infinitely many hyperbolic diagrams that arise in the context of the present paper cf. [33]; five corresponding to the graphs $\Gamma(1111), \Gamma(211), \Gamma(32)$ and the two graphs obtained by attaching a single leg of length one to the square or the triangle, plus 5 star-shaped diagram, 5 with double bonds—see the appendix—and an infinite family with just two nodes and a single higher order edge.) For example one may always take an affine ADE Dynkin diagram with dimension vector the minimal imaginary root δ, then double δ and glue a single leg of length one (with dimension one at the foot) on to the extending node, to obtain a diagram with a dimension vector for a quiver variety of dimension 4. There are other examples however, see Figure 4.

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KM root systems
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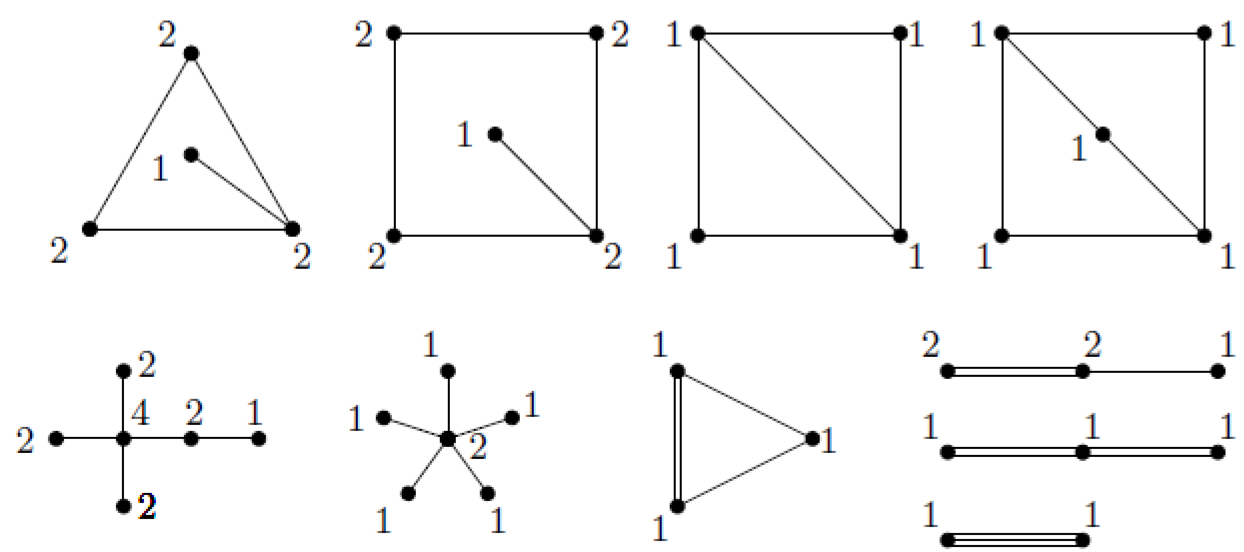

FIGURE 4. Some four dimensional cases.

Link to quiver varieties + work of Nakajima $\rightsquigarrow$

$$\mathcal{M}^*(hP^n_X) \cong \mathrm{Hilb}^n(\mathcal{M}^*(P_X)) \quad (X \neq \mathrm{III})$$

Hilbert scheme of n-points

Link to quiver varieties + work of Nakajima $\rightsquigarrow$

$$\mathcal{M}^*(hP_X^n) \cong \mathrm{Hilb}^n(\mathcal{M}^*(P_X))$$ $(X \neq \mathrm{III})$

Hilbert scheme of n-points

Conjecture Same for full moduli spaces

$$\mathcal{M}(hP_X^n) \cong \mathrm{Hilb}^n(\mathcal{M}(P_X))$$ $(\forall X)$

(and/or for Higgs bundle version)

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This has been proved recently for tame cases (D_4, E_6, E_7, E_8) by Groechenig

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so "h" can stand for <u>Higher</u>, <u>Hyperbolic</u> or <u>Hilbert</u>