Transformation groups for Isomonodromy equations

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① Painlevé equations

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$$\left(\frac{d}{dt}\right)^2 y(t) = \cdots$$

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② Schlesinger's equations

$$dA_i = -\sum_{j \neq i} [A_i, A_j] \, d\log(t_i - t_j)$$

$\underset{\sim}{t} \in \mathbb{B} = \mathbb{C}^m \setminus \text{diags}, \qquad A_i(\underset{\sim}{t}) \in \mathfrak{gl}_n(\mathbb{C})$

Two types of discrete groups appear

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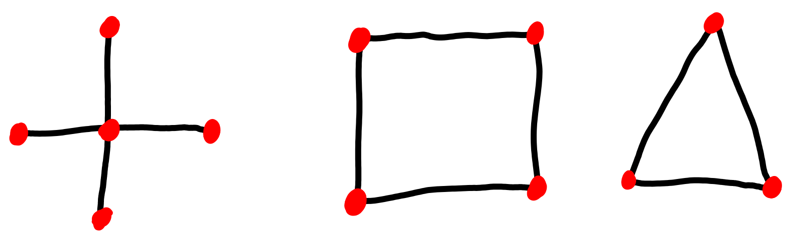
$D_4^{(1)}$ $A_3^{(1)}$ $A_2^{(1)}$ $D_2^{(1)}$ $A_1^{(1)}$ $A_0^{(1)}$



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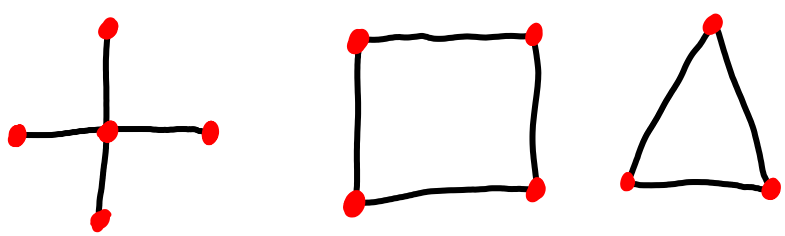

....

Geometry not immediately clear

$D_4^{(1)} \sim L\,so_8$, but $P_{VI} \sim$ IMDs of rank 2 connections with four poles on $\mathbb{P}^1$

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Malgrange/Miwa (~80): Linear connection $\sum_1^m \frac{A_i}{z-t_i}dz$ on $\mathbb{P}^1$

$\Downarrow$

Meromorphic solution of Schlesinger system on $\widetilde{\mathbb{B}}$ $\left(\mathbb{B} = \widetilde{\mathbb{B}}/P_m\right)$

E.g. P_4 : $m=3$ $P_3 \longrightarrow P_3/\mathbb{Z} = \text{Free}_2 \cong \Gamma(2) \subset PSL_2(\mathbb{Z})$

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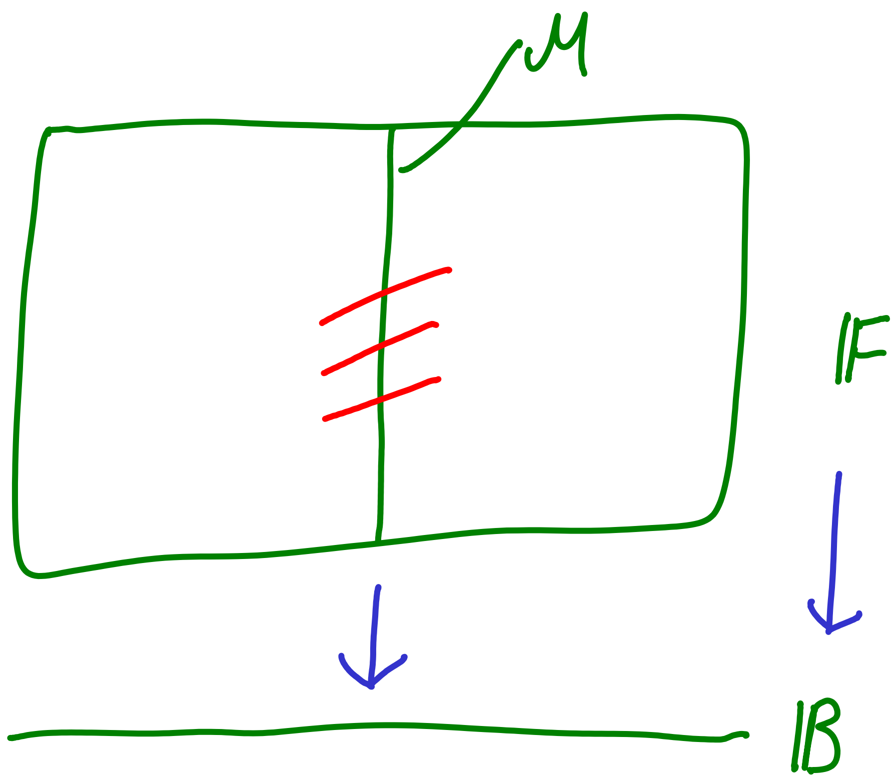
Sketch/Idea

① Rephrase isomonodromy equations as nonlinear connections on fibre bundles



(cf. PB '99, '01)

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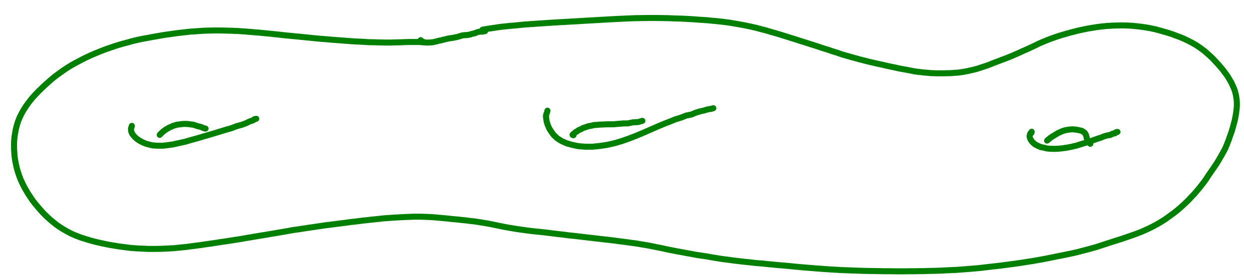
Smooth Riemann surface



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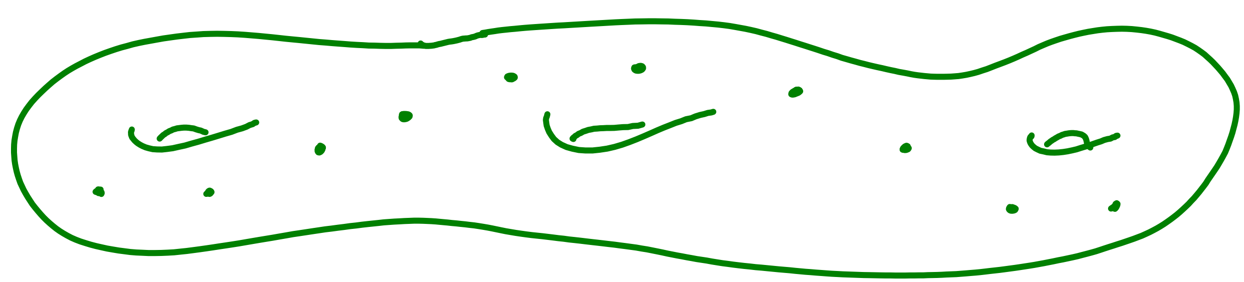
+ marked points

$\underset{\sim}{a} = (a_1, \dots, a_m)$



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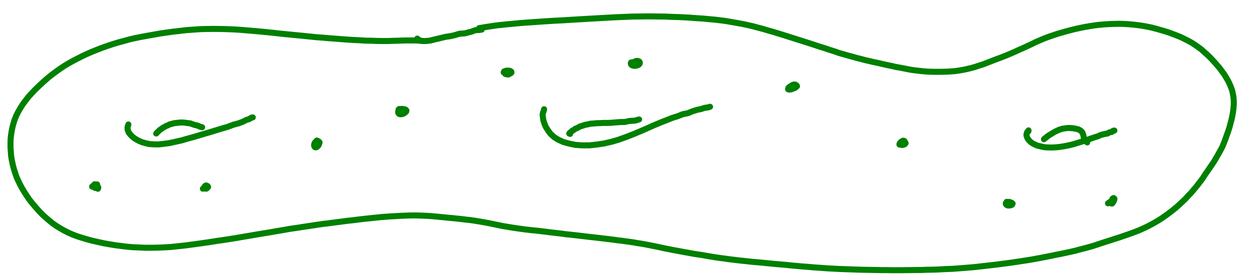

+ an "irregular type" $Q_i = \frac{A_{r_i}}{z^{r_i}} + \cdots + \frac{A_1}{z}$ at each a_i $\left\{ \begin{array}{l} A_i \in \mathfrak{t} \\ z(a_i) = 0 \end{array} \right.$

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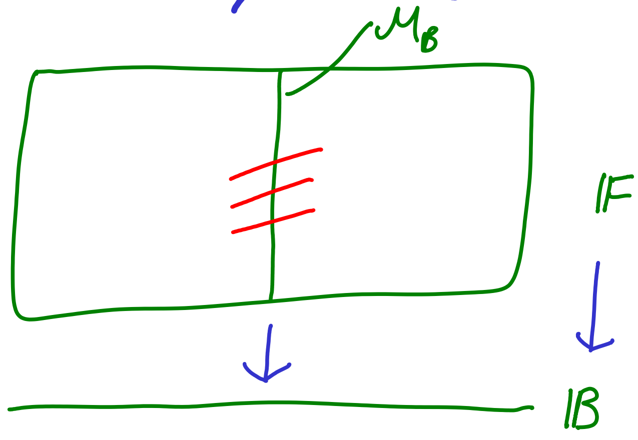

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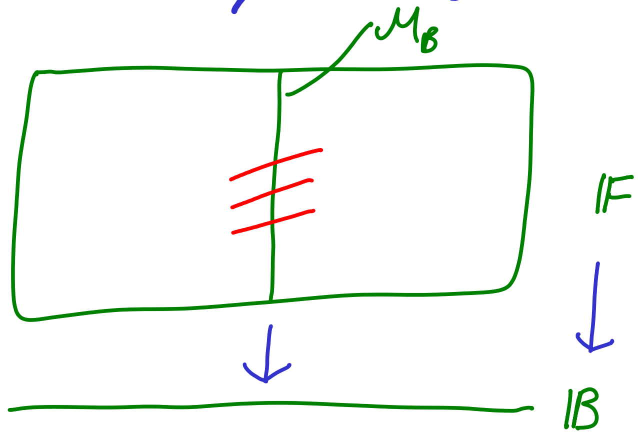

$\Rightarrow$ integrates to give nonlinear monodromy action

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(generalises Hurwitz braid action & q-c qWeyl)

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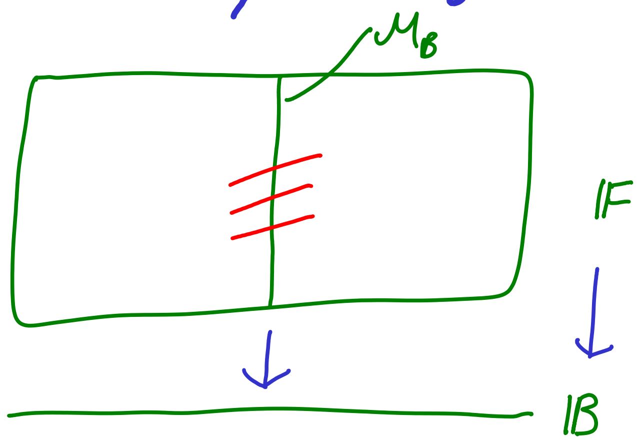

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⑤ Conjugate by Riemann-Hilbert-Birkhoff to get isomonodromy connection

$$\begin{array}{ccccccc} \mathcal{M}_{DR}(\Sigma_b) & \hookrightarrow & \mathbb{F}_{DR} & \xrightarrow[\cong]{RHB} & \mathbb{F}_B & \hookleftarrow & \mathcal{M}_B(\Sigma_b) \\ & & \downarrow & & \downarrow & & \\ & & \mathbb{B} & = & \mathbb{B} & & \end{array}$$

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Three levels: ① Nonlinear connection

② Matrix equations (e.g. Schlesinger, JMMS, JMU)

③ Scalar equations (e.g. Painlevé equations)

– each has strengths and weaknesses

	S	W
1	coord indep't.	hard to teach to undergraduates
2	good balance explicit vs generality	(can get complicated)
3	as explicit as possible	many equivalent expressions not known/too messy in many cases

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- e.g. JMU's injective RHB map is $\mathcal{M}^* \hookrightarrow \mathcal{M}_B$
& it factors as $\mathcal{M}^* \subset \mathcal{M}_{DR} \xrightarrow[\text{RHB}]{\cong} \mathcal{M}_B$ (P.B '99,'01)

Irregular connections

Local formal classification: (Hukuhara–Turrittin)

After passing to a finite cover any meromorphic connection on a vector bundle on a curve is formally meromorphically isomorphic to

$$D = d - \left(dQ + \Lambda \frac{dz}{z} \right)$$

where
- $Q = \frac{A_r}{z^r} + \cdots + \frac{A_1}{z}$, A_i diagonal $n \times n$ matrices
- $\Lambda \in \mathrm{End}(\mathbb{C}^n)$ commutes with $A_1, \dots, A_r$

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"Irregular type" "formal residue"

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Can describe ? in terms of fundamental groupoid of a related curve ...

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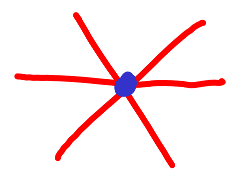

- $\exists$ finite no. of singular directions
- For any non-singular direction d $\exists$ preferred analytic isomorphism $F_d = \mathrm{Sum}_d(F)$ relating B & A^0

Irregular Betti spaces

Let Σ be an irreg. curve (marked points $a_1, \ldots, a_m$, irreg. types $Q_1, \ldots, Q_m$)

Let $\hat{\Sigma} \longrightarrow \Sigma$ be real oriented blow up of Σ at a_i

(each a_i replaced by a circle ∂_i, so $\partial \hat{\Sigma} = \partial_1 \sqcup \ldots \sqcup \partial_m$)

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Then each Q_i determines:

1) A connected complex reductive group $H_i \subset G$

2) A finite set $\mathbb{A}_i \subset \partial_i$ of singular directions at a_i

and for each $d \in \mathbb{A}_i$

3) A unipotent group $\mathrm{Sto}_d(Q_i) \subset G$ normalised by H_i

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so $\mathfrak{g} = t \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid [Y, X] = \alpha(Y) X \ \forall Y \in t \}$

Let $q_\alpha = \alpha \circ Q$ (mero. function near $a \in \Sigma$)

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then $d \in \partial$ is a singular direction supported by $\alpha \in \mathcal{R}$

if $\exp(q_\alpha)$ has maximal decay as $z \to a$ along d

(leading term of q_α is real and negative along d)

& $\mathbb{A} \subset \partial$ is set of all sing. directions ($\forall \alpha \in \mathcal{R}$)

3) Let $R(d) = \{ \alpha \mid \alpha \text{ supports } d \} \subset R$

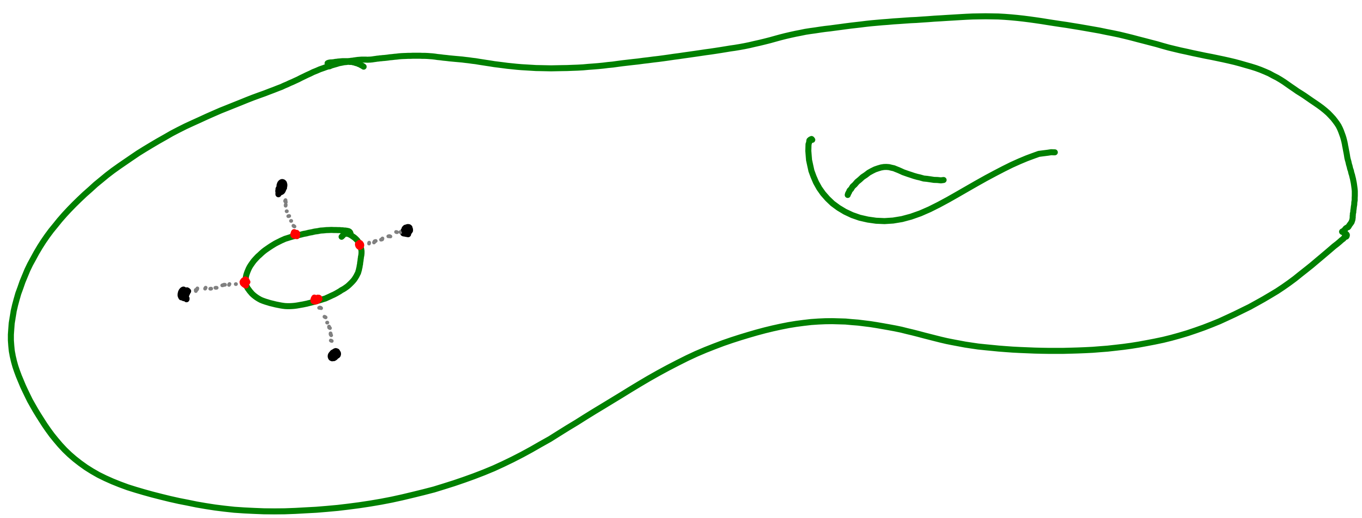
$$St_d = \prod_{\alpha \in R(d)} \exp(\mathfrak{g}_\alpha) \hookrightarrow G$$

Lemma St_d is a well defined unipotent subgroup of G

Now puncture $\hat{\Sigma}$ once in its interior near each singular direction $d \in \mathbb{A}_i$, $i = 1, \dots, m$

and let $\tilde{\Sigma} \subset \hat{\Sigma}$ be resulting punctured surface

$\tilde{\Sigma}$



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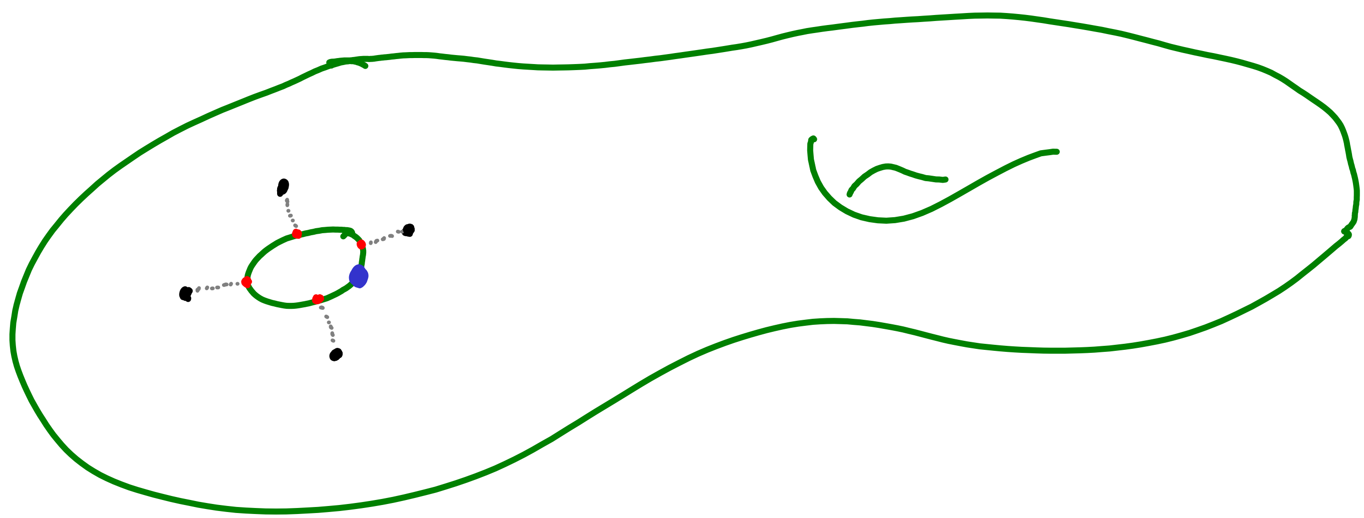

Choose a base point $b_i \in \partial_i$ in each boundary circle

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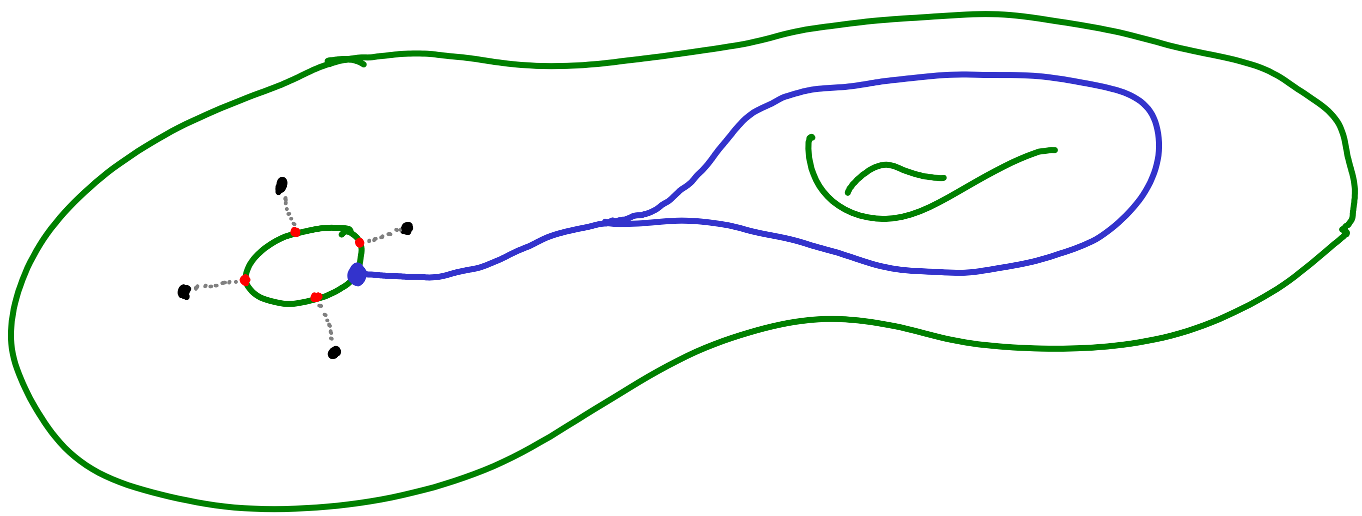

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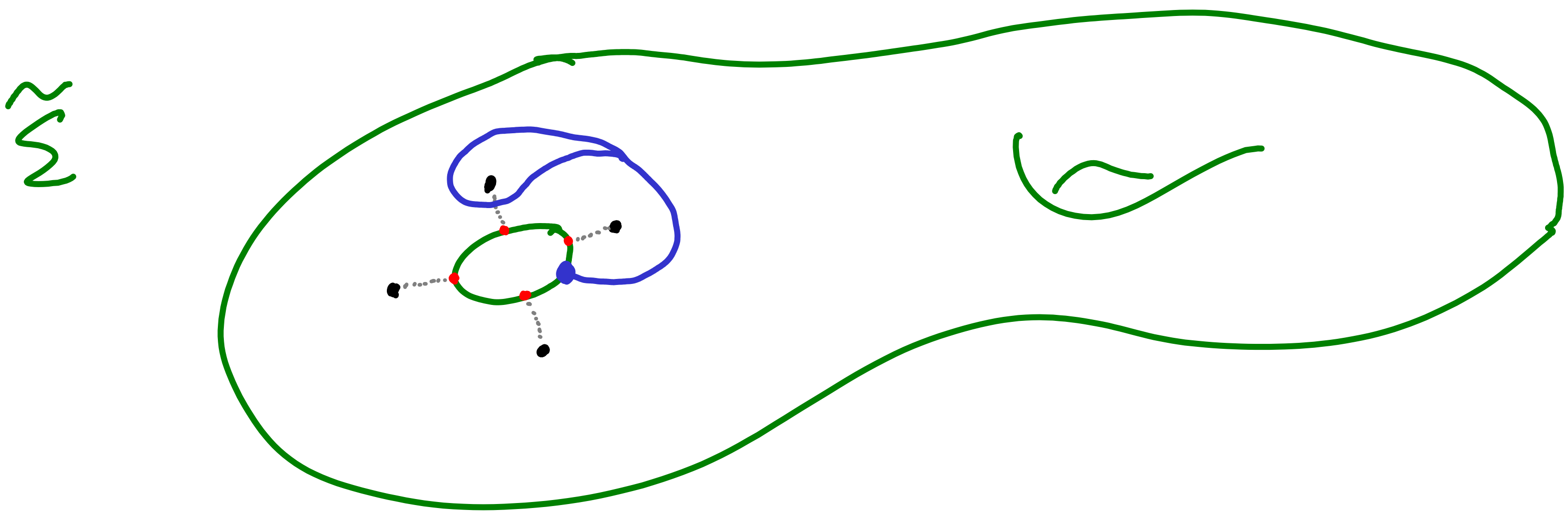

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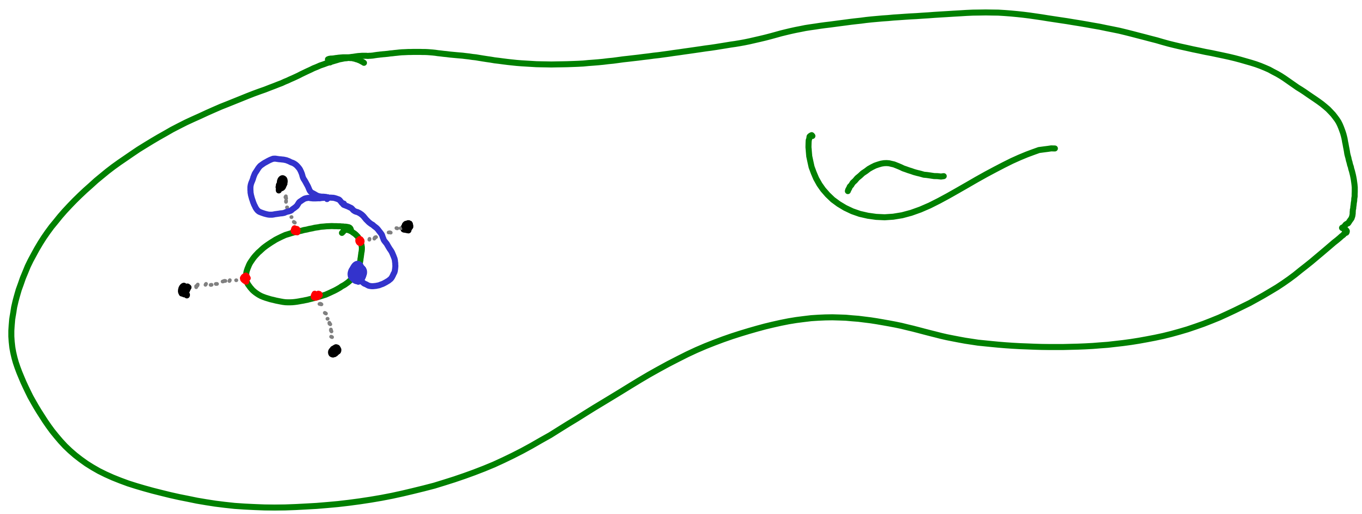

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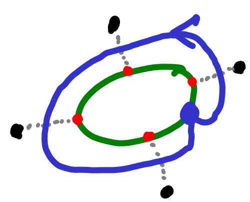

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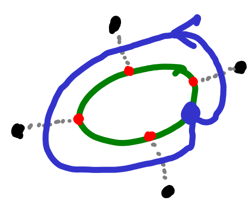
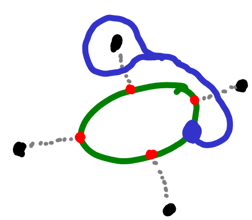
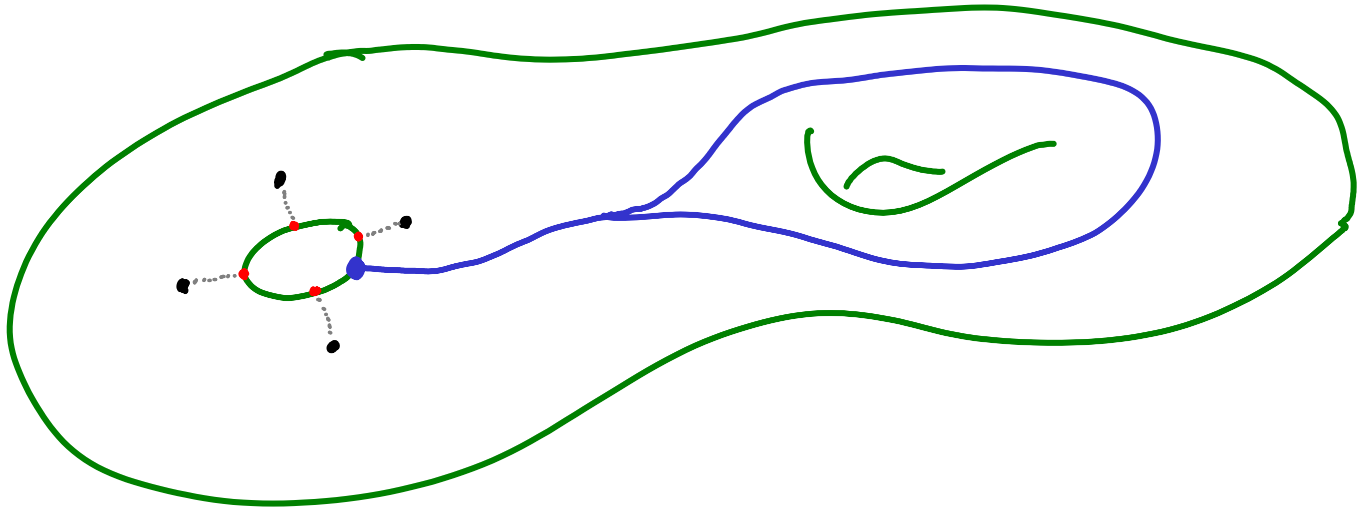


2) If γ goes around ∂_i from b_i until $d \in \mathbb{A}_i$ then loops around the corresponding puncture before returning to b_i, then $\rho(\gamma) \in \mathbb{S}to_d$



Connections with given irreg. types $\rightsquigarrow$ Stokes representations

$\tilde{\Sigma}$



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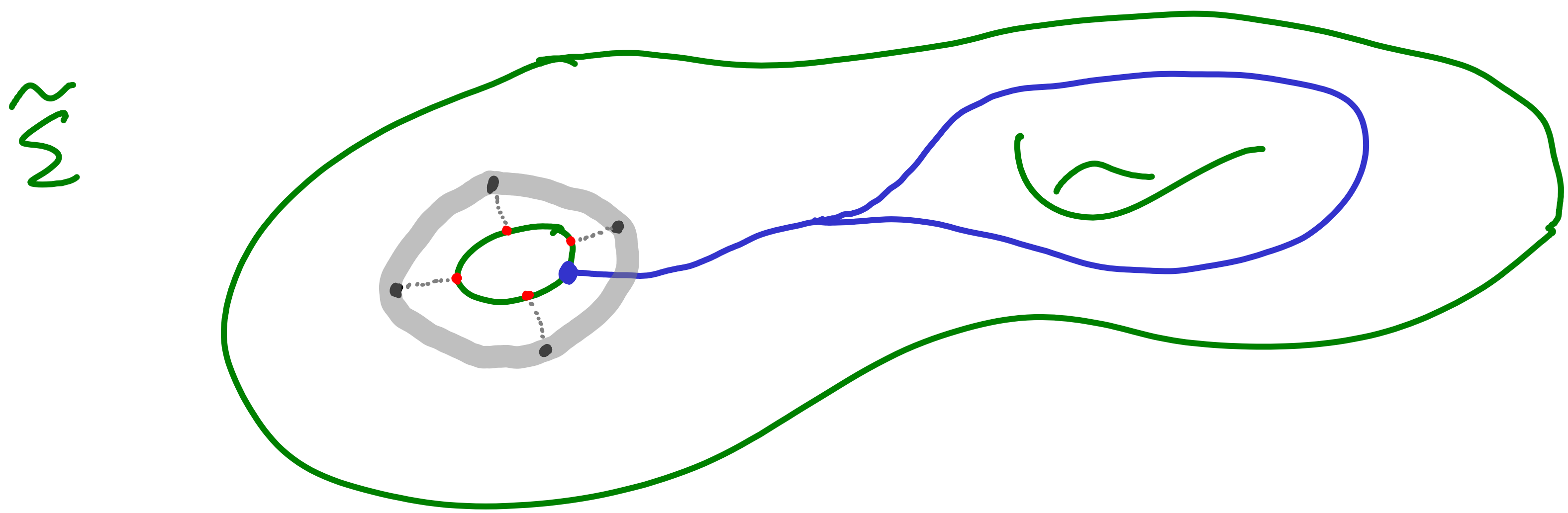

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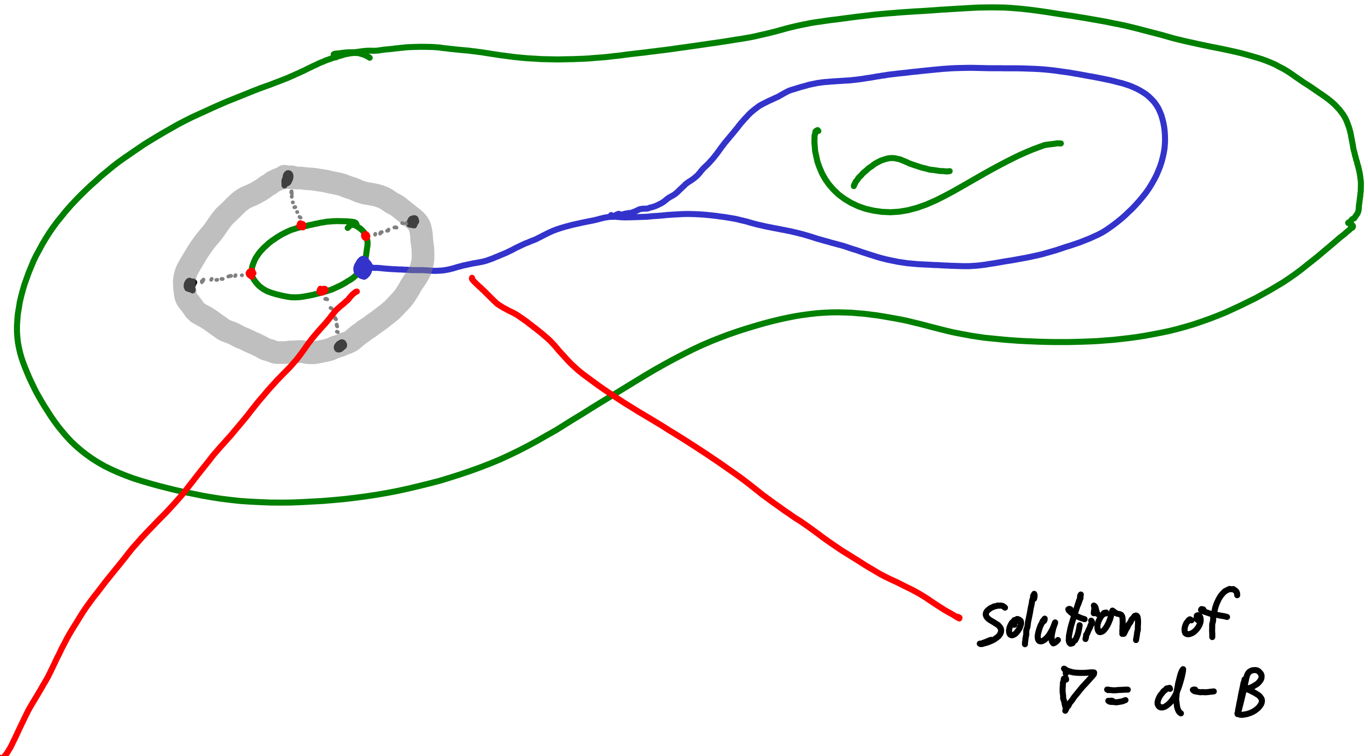

Solution of $\nabla = d - B$

Solution of $A^0 = dQ + \Lambda \frac{dz}{z}$

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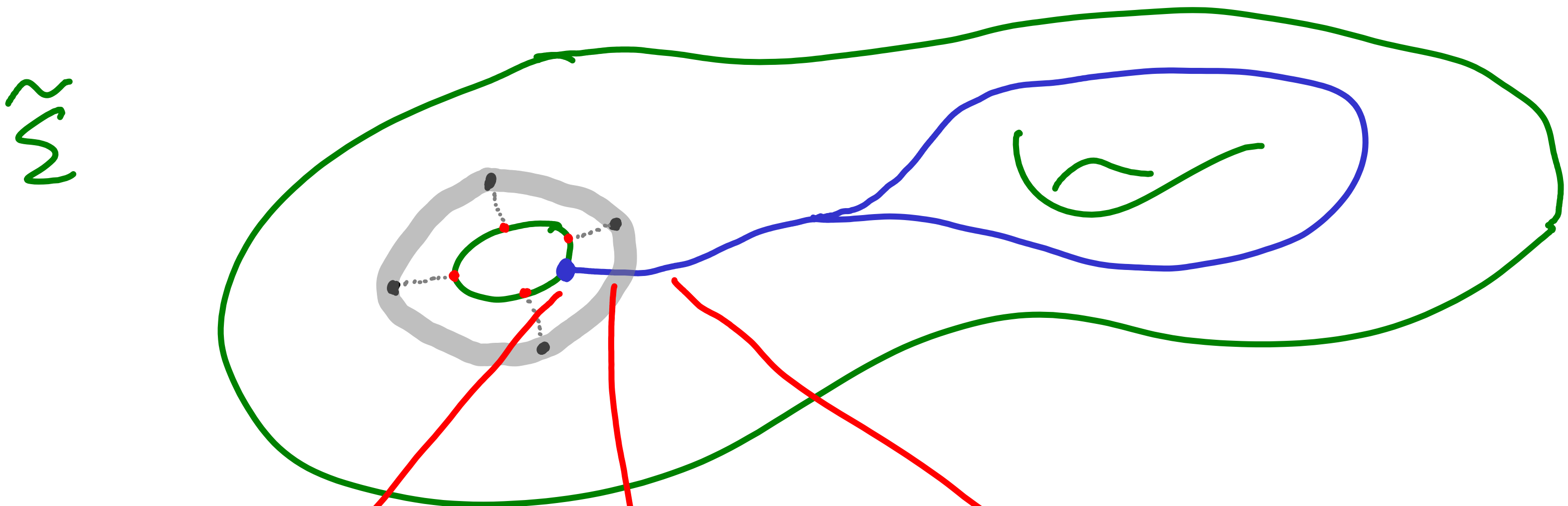

Solution of
$\nabla = d - B$

Solution of
$A^0 = dQ + \Lambda \frac{dz}{z}$

glue solutions via F_d
(between singular directions)

Theorem (building on Birkhoff Balser Jurkat Lutz Malgrange Sibuya Deligne Martinet Ramis ...)

$$\left\{ (V,\nabla) \;\middle|\; \begin{array}{l} V \to \Sigma^{\circ} \text{ alg. vector bundle} \\ \nabla \text{ alg. connection s.t.} \\ \text{irreg. type } Q_i \text{ at } a_i \end{array} \right\} \Big/ \text{isom.} \quad \xleftrightarrow[\cong]{\text{Riemann-Hilbert-Birkhoff}} \quad \mathrm{Hom}_{\mathbb{S}}(\Pi, G) / \underset{\sim}{H}$$

where $\underset{\sim}{H} = H_1 \times \cdots \times H_m$

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- Have notion of irreducible Stokes representations:

Symp. leaves of $\mathrm{Hom}^{irr}_{\mathbb{S}}(\Pi, G)/\underset{\sim}{H}$ are hyperkähler (Biquard–B. 2004)

and diffeomorphic to spaces of meromorphic Higgs bundles

$$
\begin{tikzcd}
 & & \mathcal{M}_{Dol} \\
\Sigma \arrow[r, Rrightarrow] & \mathcal{M}(\Sigma) \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{M}_{DR} \\
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\end{tikzcd}
$$

irregular curve

Hyperkahler manifold

meromorphic Higgs bundles

mero. Hitchin systems

(Bottacin - Markman)

mero. connections

Stokes representations

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Σ: irregular curve

$\mathcal{M}(\Sigma)$: Hyperkähler manifold

$\mathcal{M}_{Dol}$: meromorphic Higgs bundles; mero. Hitchin systems (Bottacin - Markman)

$\mathcal{M}_{DR}$: mero. connections

$\mathcal{M}_{Betti}$: Stokes representations

Can define "admissible deformations" of Σ

& obtain generalisation of braid & mapping class group actions

on $\mathcal{M}_{Betti}$ (Jimbo–Miwa–Ueno '81, B. '01, '11)

Suppose

$$\begin{tikzcd} \Sigma_b \arrow[r, hook] & \Sigma \arrow[d] \\ & \mathbb{B} \end{tikzcd}$$

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Σ is an "admissible deformation" of Σ_b if

- Each fibre $\Sigma_t = \pi^{-1}(t) \subset \Sigma$ is smooth $(t \in \mathbb{B})$

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Σ is an "admissible deformation" of Σ_b if

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- points $a_1, \dots, a_m$ remain distinct
- for any root $\alpha : \mathfrak{t} \to \mathbb{C}$ & any irregular type Q_i

$$\mathrm{PoleOrder}(\alpha \circ Q_i) \in \{0, 1, 2, \dots\}$$

remains constant

Theorem If
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of wild character varieties / irregular Betti spaces assembles into a local system of Poisson varieties:

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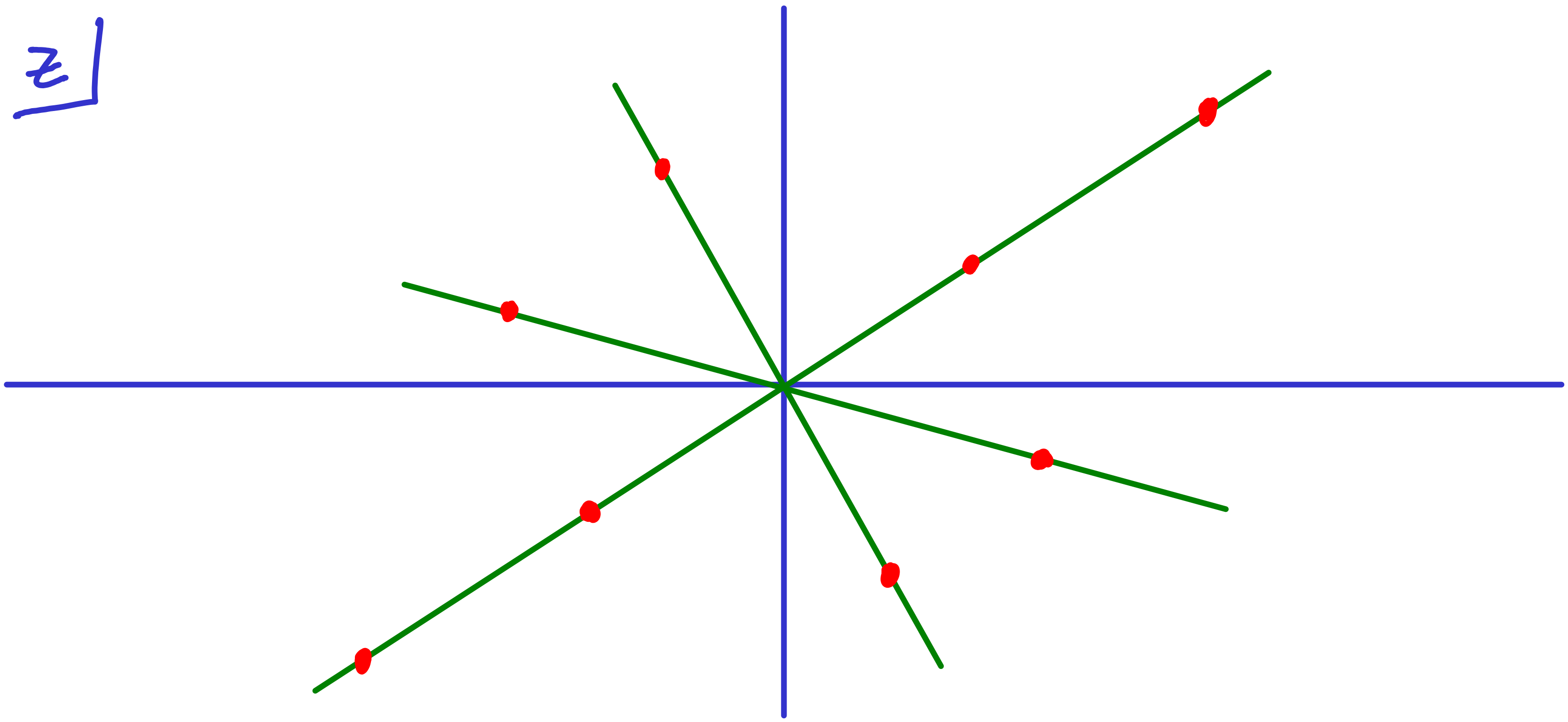
$$\boxed{\begin{array}{c}\Rightarrow \pi_1(\mathbb{B}) \circlearrowright \mathcal{M}_B(b) \\ \text{via algebraic Poisson autom.s} \\ \text{(irreg. mapping class group action)}\end{array}}$$

– nonlinear fibration with complete, flat (Ehresmann) connection

Simplest example $\quad r = 1, \quad Q = \frac{-A_1}{z}, \quad A_1 \in t_{reg}$

Plot roots on z-plane: $\langle A_1, R \rangle \subset \mathbb{C}^*$



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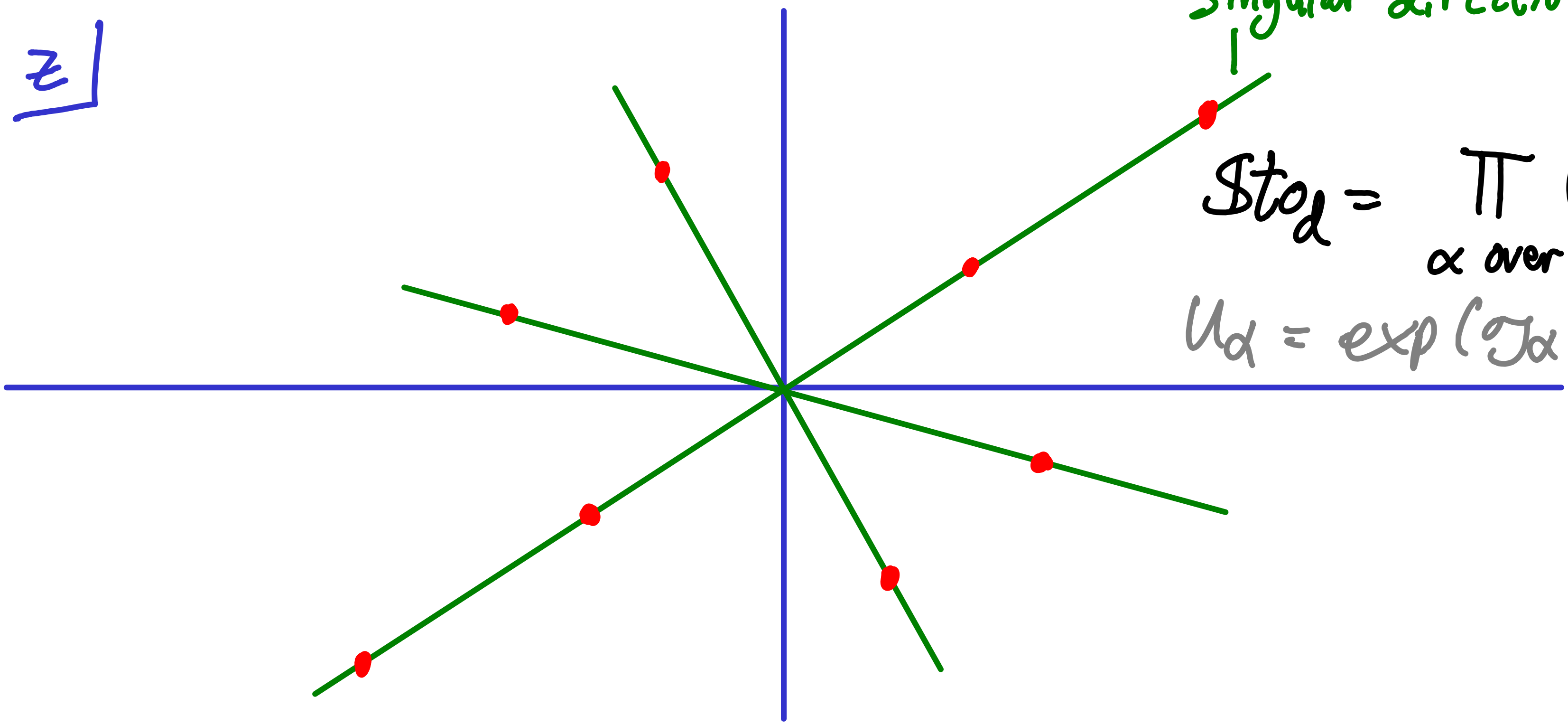

$$\mathbb{S}to_d = \prod_{\alpha \text{ over } d} U_\alpha \subset G$$

$U_\alpha = \exp(\mathfrak{g}_\alpha)$ root gp

$$\{\text{Stokes data}\} = \prod_d \mathbb{S}to_d$$

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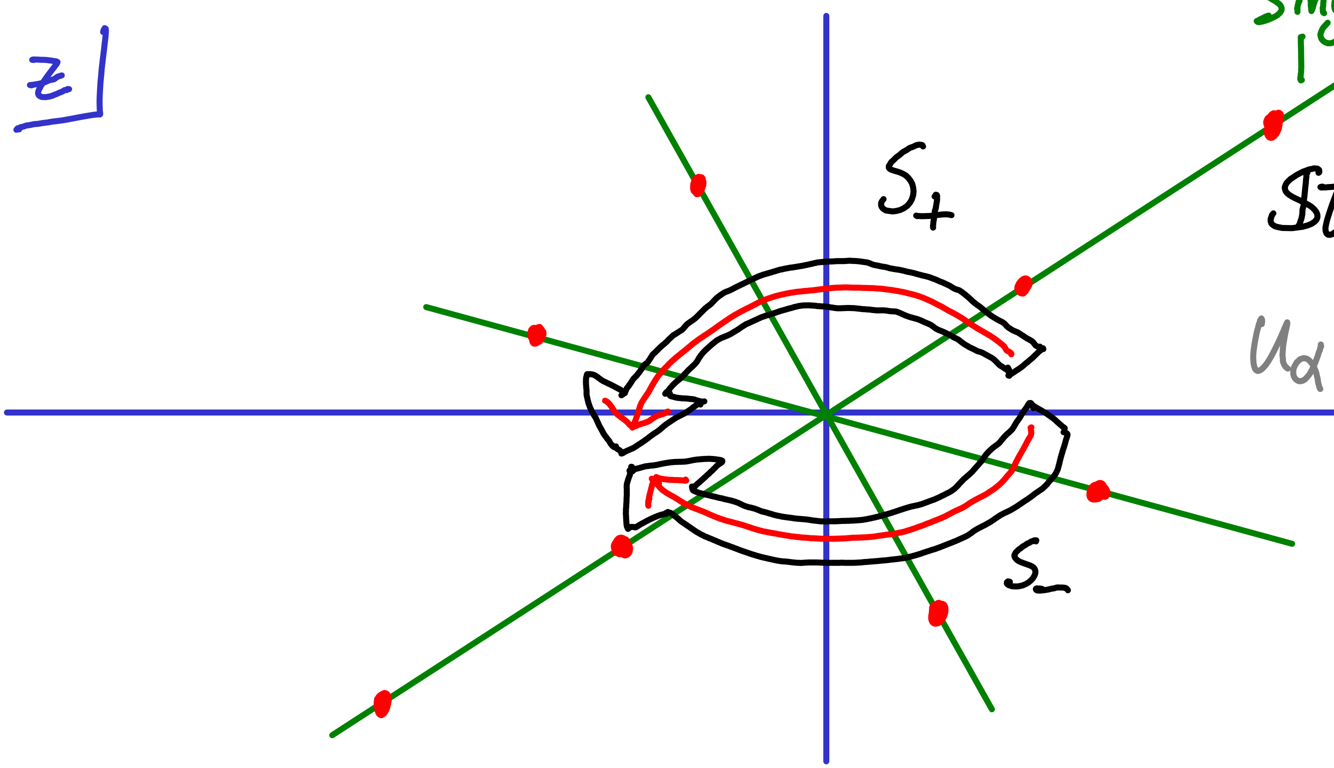

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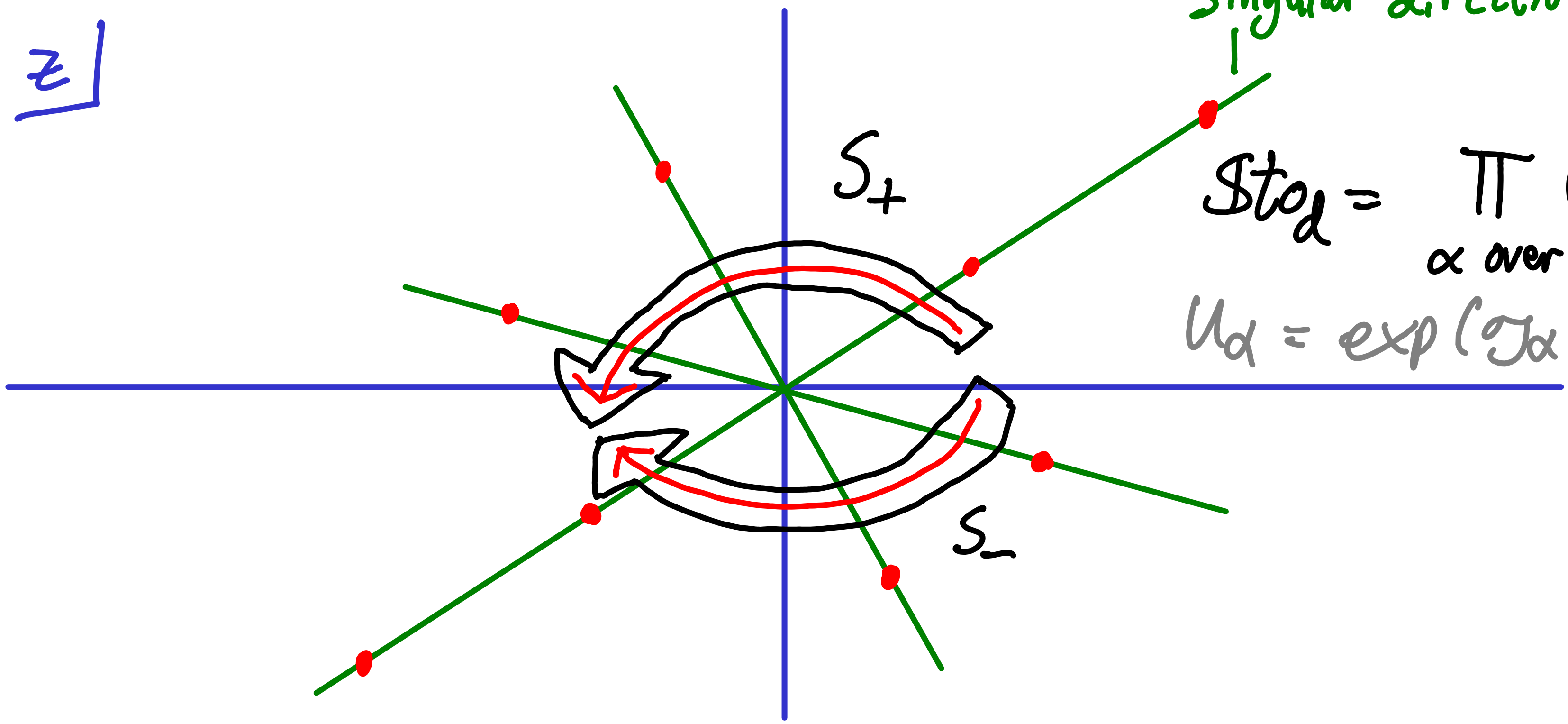

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Isomonodromy: Vary $A_1 \in \mathfrak{t}_{reg}$ & keep $S_\pm$ const. (locally)

In this example the resulting braid gp action had been previously seen:

$$\begin{tikzcd}
& U_q\mathfrak{g} & \\
\mathrm{Fun}(G^*) \arrow[ur] & & U\mathfrak{g} \arrow[ul] \\
& \mathrm{Fun}(\mathfrak{g}^*) = \mathrm{Sym}(\mathfrak{g}) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$G^* \cong T \ltimes (U_+ \times U_-)$ dual Poisson-Lie group (Drinfeld)

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$B_{\mathfrak{g}} \circlearrowleft U_q\mathfrak{g}$ Quantum Weyl group action (Lusztig, Soibelman ...)

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DeConcini-Kac-Procesi $\Downarrow$

Quantum Weyl group action (Lusztig, Soibelman ...)

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Thm (- '02)

The DKP action arises from isomonodromy ($U_+ \times U_-$ = Stokes data)

- Purely geom. origin (not just explicit generators)
- $U_q\mathfrak{g}$ quantizes a moduli space of mero. connections