Weyl group transformations

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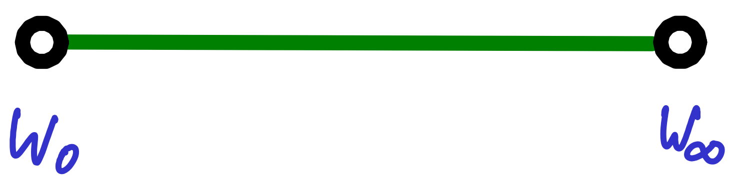
Revisit the JMMS equations (Jimbo-Miwa-Mori-Sato 1980)

Choose two finite dimensional vector spaces W_0, W_∞



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$$\begin{tikzcd} W_0 \arrow[r, bend left, "P"] & W_\infty \arrow[l, bend left, "Q"] \end{tikzcd}$$

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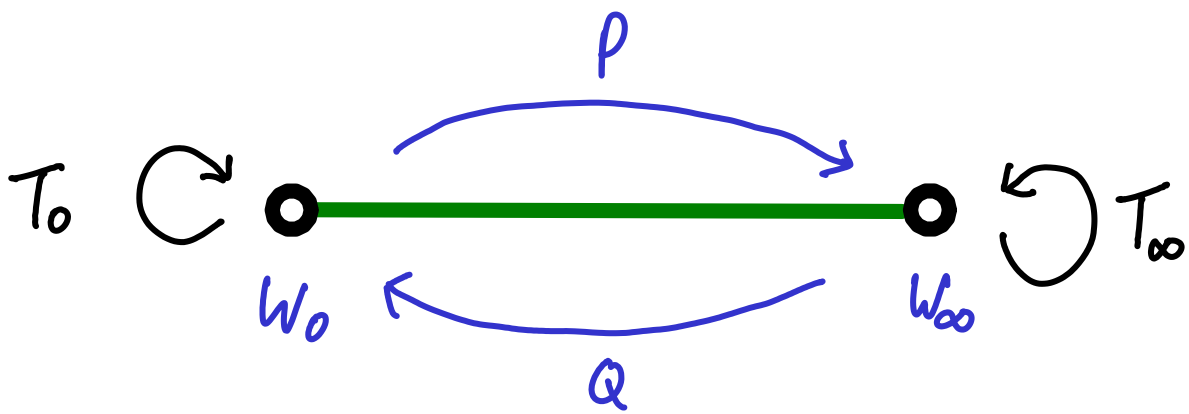
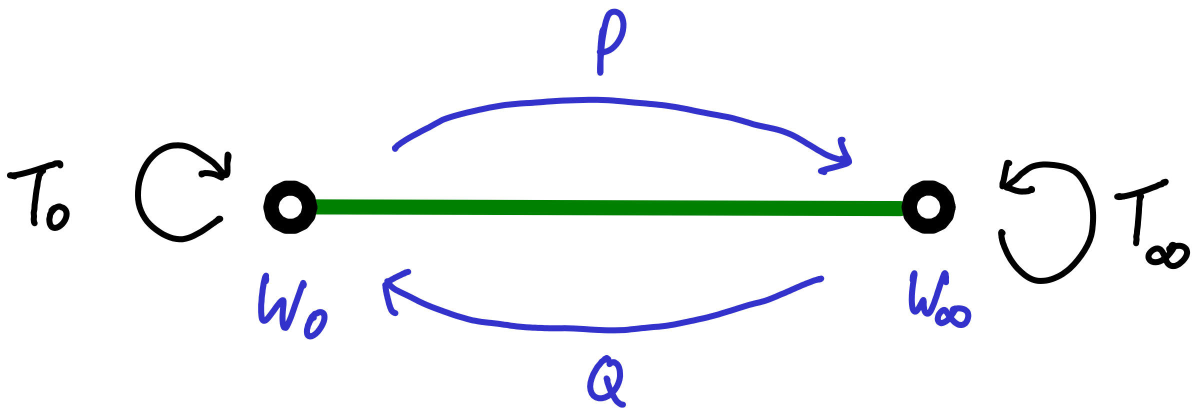


T_i diagonalisable; Eigenvalues of T_i are the times

– no further coalescences permitted

– eigen spaces fixed



Can write JMMS equations as follows:

$$dQ = Q\,\widetilde{PQ} + \widetilde{QP}\,Q + T_0\,Q\,dT_\infty + dT_0\,Q\,T_\infty$$

$$-dP = P\,\widetilde{QP} + \widetilde{PQ}\,P + T_\infty\,P\,dT_0 + dT_\infty\,P\,T_0$$

$$
\begin{tikzcd}
W_0 \arrow[loop left, "T_0"] \arrow[r, bend left, "P"] & W_\infty \arrow[loop right, "T_\infty"] \arrow[l, bend left, "Q"]
\end{tikzcd}
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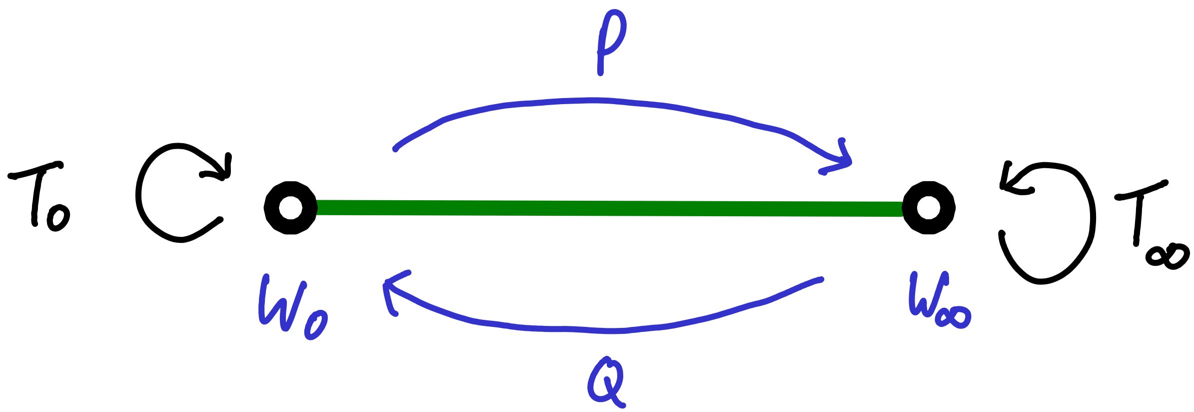
where $\tilde{R} = \mathrm{ad}_{T_i}^{-1}[dT_i, R]$ for $R \in \mathrm{End}(W_i)$

$\left(\tilde{R}_{ab} = R_{ab}\, d\log(t_a - t_b) \quad \text{if } T_i = \sum t_a \mathrm{Id}_a \right)$



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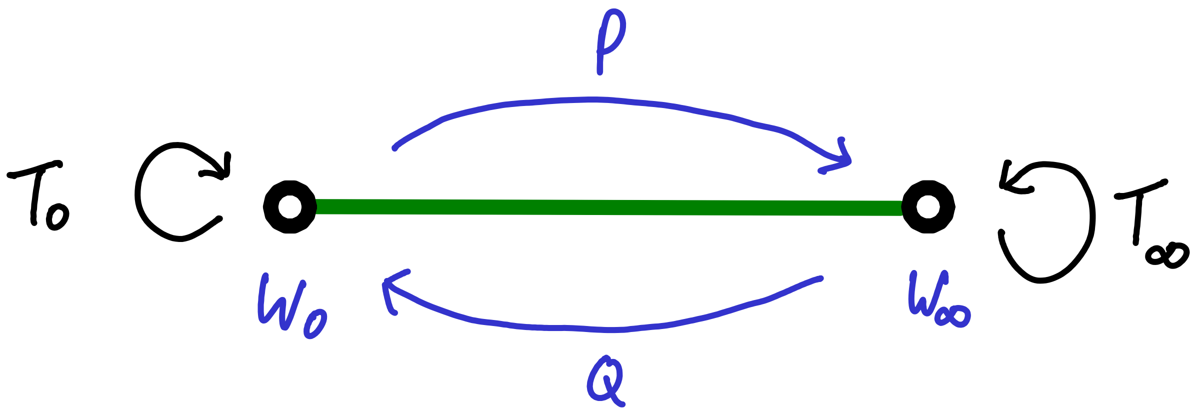
E.g. $T_0 = 0$ JMMS $\Leftrightarrow$ Schlesinger equations



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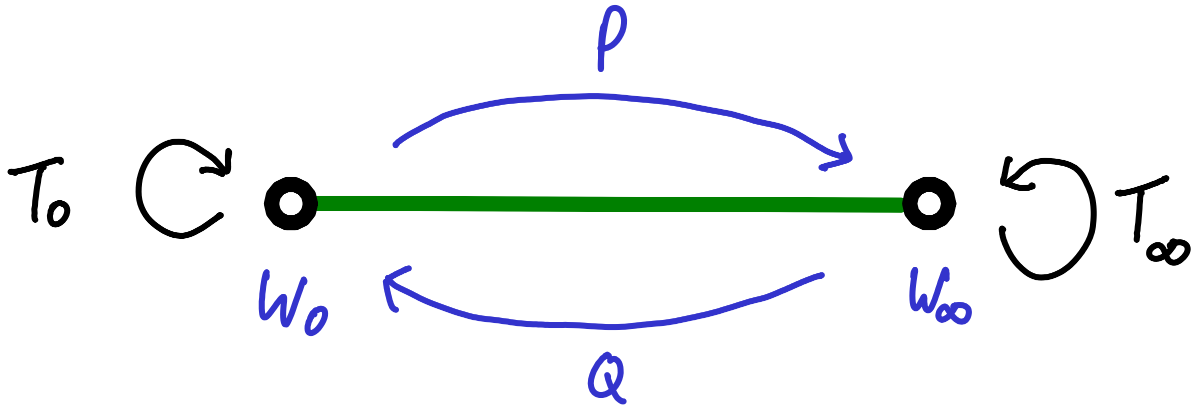
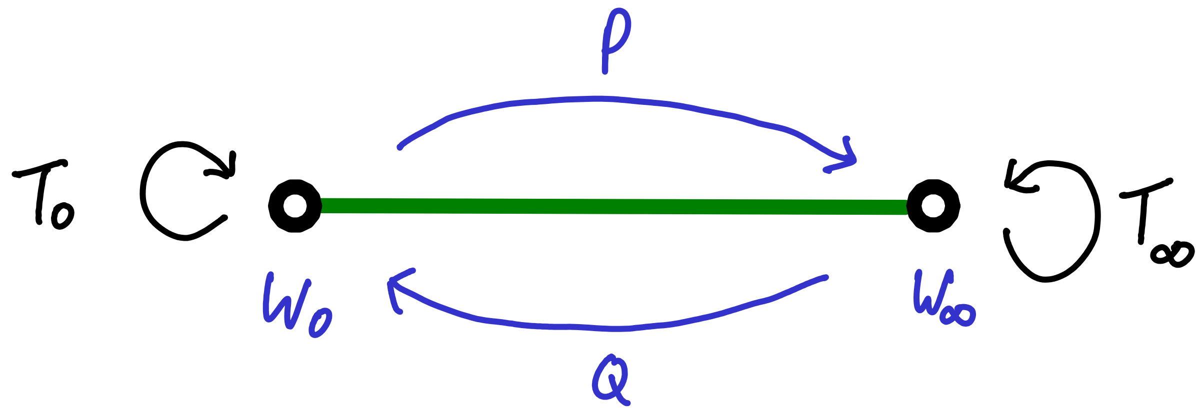
Thm (Harnad '94)

The permutation $(W_0, W_\infty, P, Q, T_0, T_\infty) \longmapsto (W_\infty, W_0, Q, -P, -T_\infty, T_0)$ preserves the JMMS equations



Harnad's duality $(W_0, W_\infty, P, Q, T_0, T_\infty) \longmapsto (W_\infty, W_0, Q, -P, -T_\infty, T_0)$ basically flips over the graph.



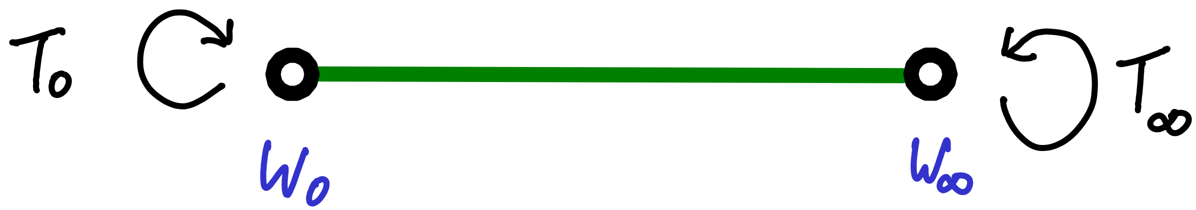
Harnad's duality $(W_0, W_\infty, P, Q, T_0, T_\infty) \longmapsto (W_\infty, W_0, Q, -P, -T_\infty, T_0)$
basically flips over the graph.

JMMS system controls isomonodromic deformations of

$$\left(T_0 \quad + \quad Q\,(z - T_\infty)^{-1} P\right) dz \qquad \text{on} \qquad W_0 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

so $\quad W_0 \leftrightarrow W_\infty \quad$ changes rank of the vector bundle

Splaying / additive fission



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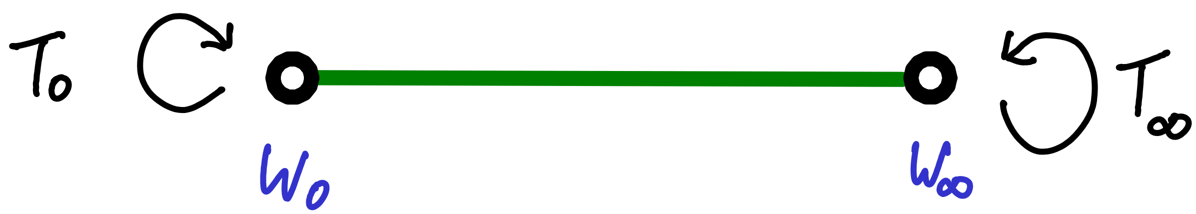

W_0, W_∞ decompose into eigenspaces of T_0, T_∞:

$$W_j = \bigoplus_{i \in I_j} V_i \qquad (I_0,\ I_\infty \text{ label eigenspaces})$$

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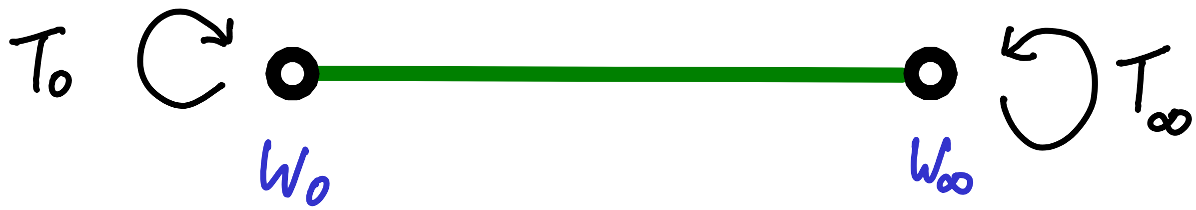

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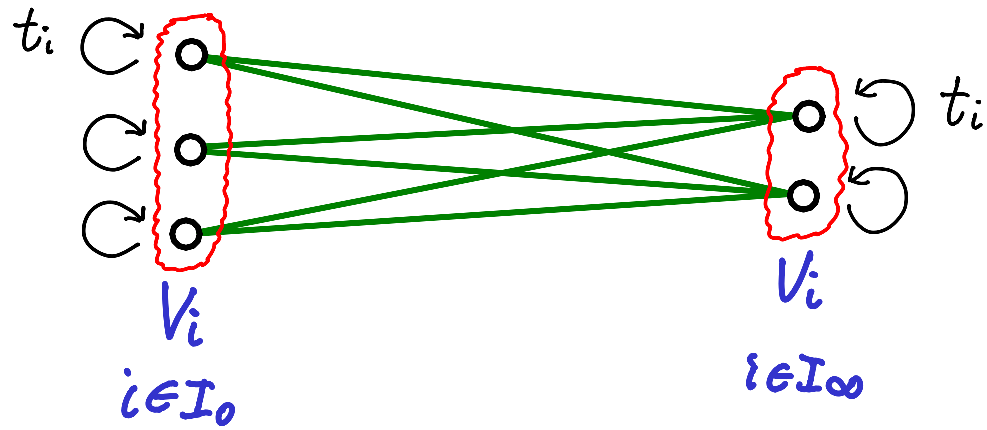
$$T_j = \sum_{i \in I_j} t_i \, \mathrm{Id}_i$$

$$\begin{cases} t_i \in \mathbb{C} & \text{eigenvalues/times} \\ \mathrm{Id}_i = \mathrm{Id}_{V_i} \in \mathrm{End}(W_j) \end{cases}$$

Splaying / additive fission



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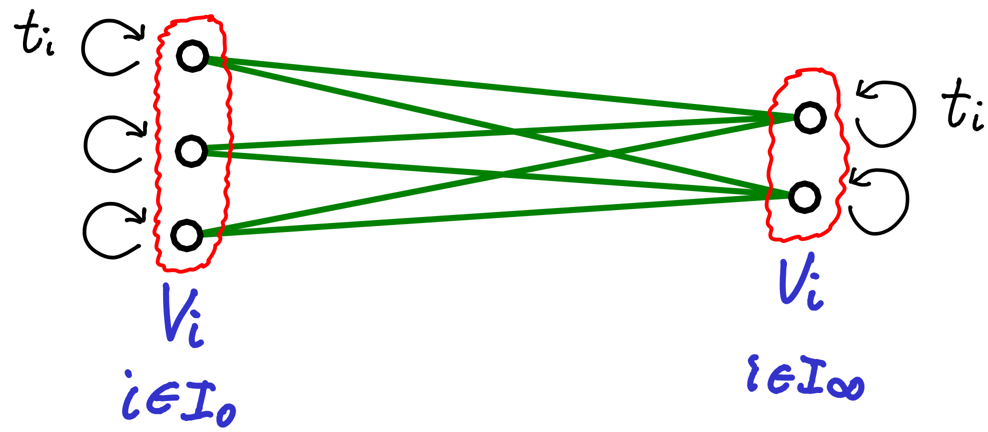
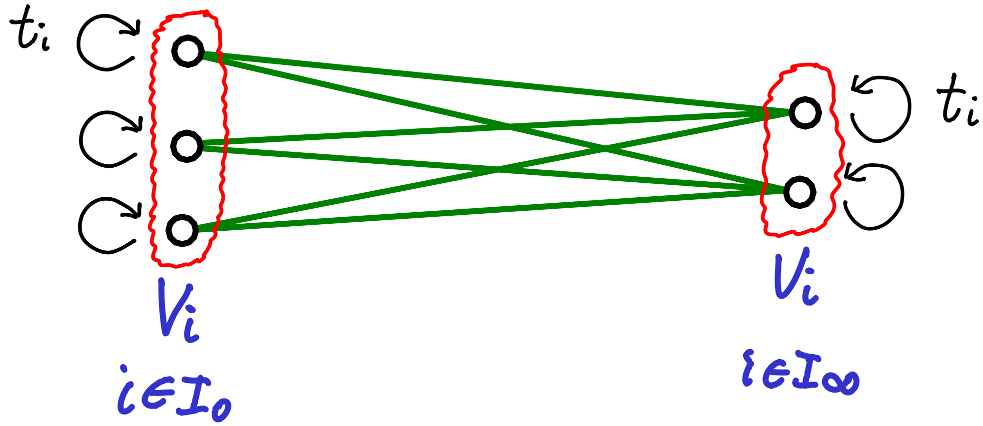


Dependent variables P, Q decompose:

nodes $I = I_0 \sqcup I_\infty$

$$\begin{array}{c}(P, Q) \\ \cap \\ \mathrm{Hom}(W_0, W_\infty) \oplus \mathrm{Hom}(W_\infty, W_0)\end{array} \iff p_{ij} : V_j \to V_i \quad \forall i, j \in I \text{ s.t. } \exists \text{ edge } i - j$$

Splaying / additive fission



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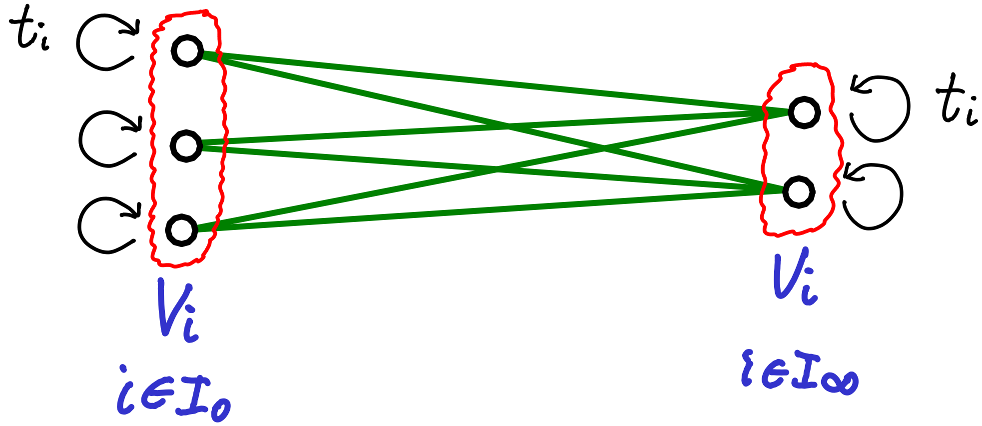
"Representation of the graph on $V = \bigoplus_{i \in I} V_i$"

Splaying / additive fission

All complete bipartite graphs arise for the JMMS equations:

$\mathcal{G}(3,2) \cong$



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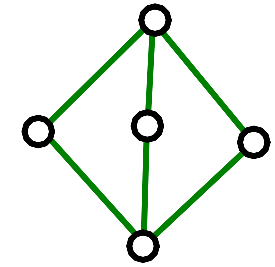

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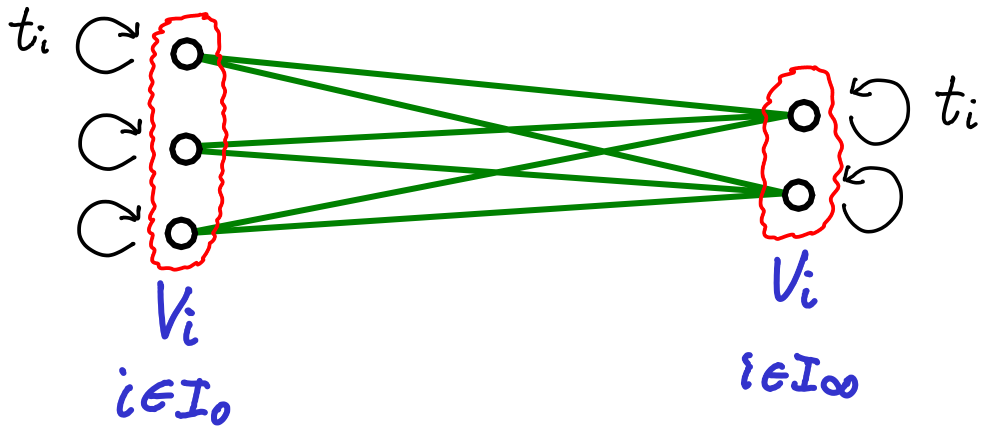

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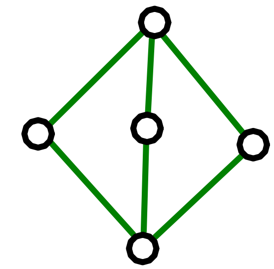

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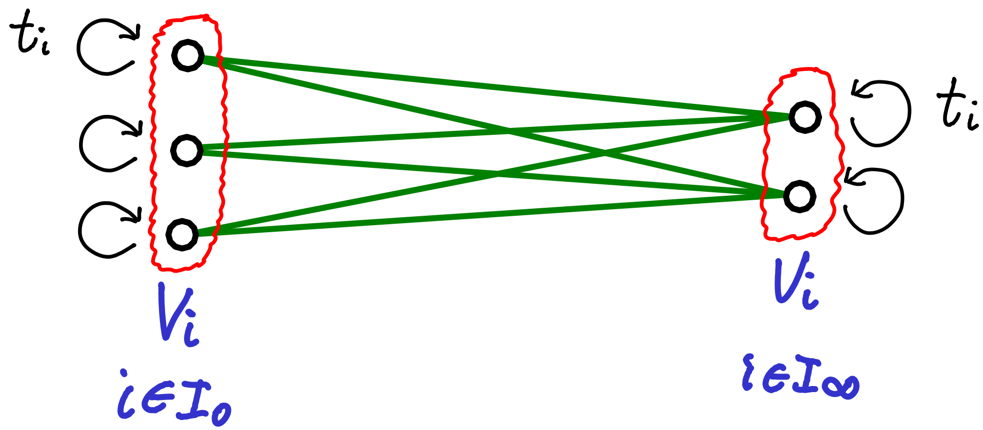

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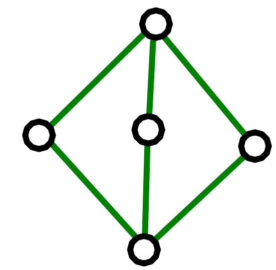
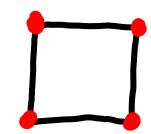


Observe: 1 (JMMS '80) If $|I_0| = |I_\infty| = \dim W_0 = \dim W_\infty = 2$

then JMMS equations $\Longleftrightarrow$ Painlevé V

2 (Okamoto '85) Painlevé V has $A_3^{(1)}$ symmetry



3 $\mathcal{G}(2,2)$ is a square

Splaying / additive fission

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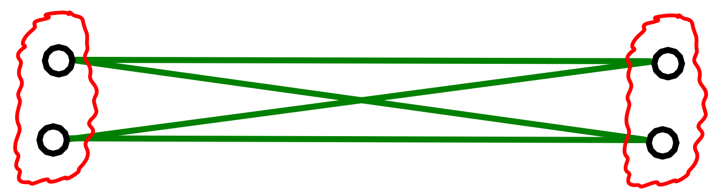
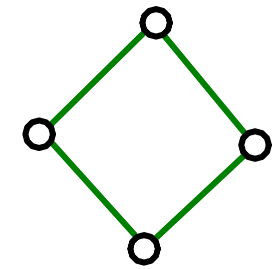
$\mathcal{G}(2,2) \cong$  $\cong$ 

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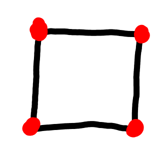
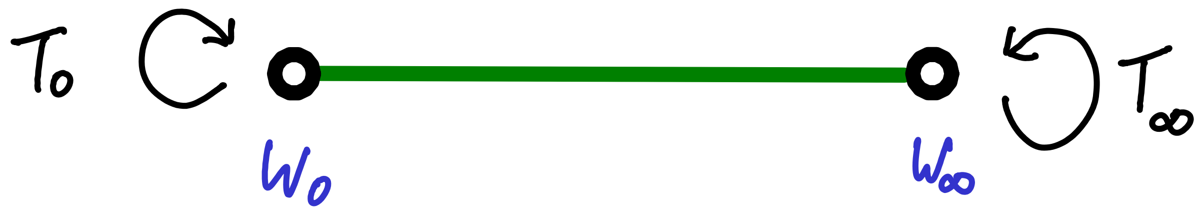
② (Okamoto '85) Painlevé V has $A_3^{(1)}$ symmetry 

③ $\mathcal{G}(2,2)$ is a square

Harnad duality (+ Schles. trsms) $\Rightarrow$ Okamoto syms

Generalisation:

Replace initial graph



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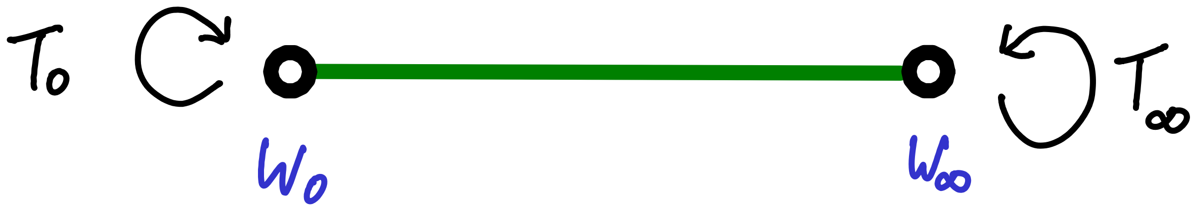
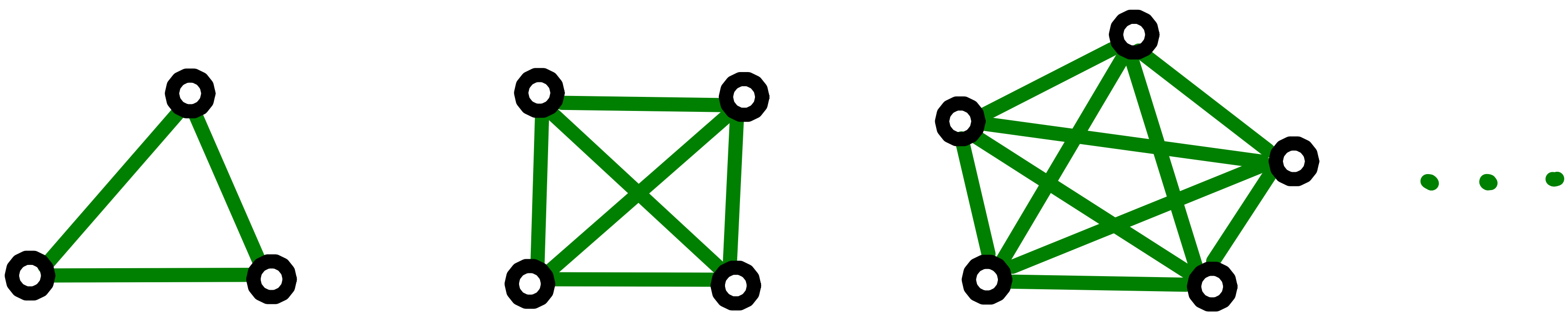


by an arbitrary complete graph:



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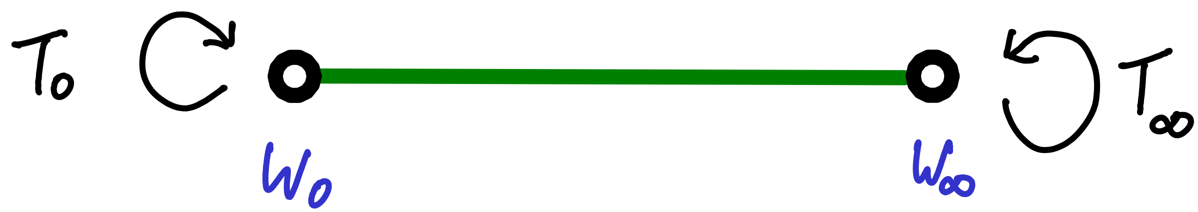

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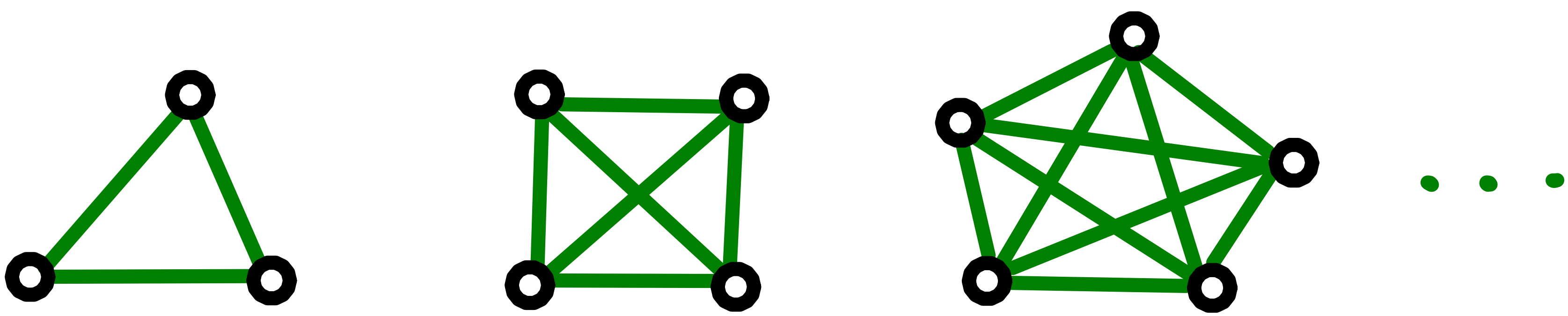

Label nodes by points $\mathcal{J} = \{a_j\} \hookrightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$

Put vector spaces W_j at nodes $(\forall j \in \mathcal{J})$, & "times" $T_j \in \mathrm{End}(W_j)$ (diagonalisable)

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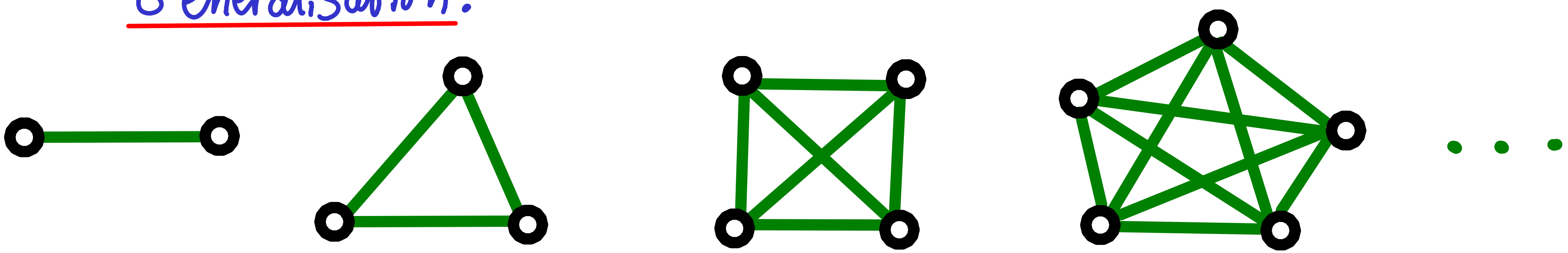

Label nodes by points $J = \{a_j\} \hookrightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$

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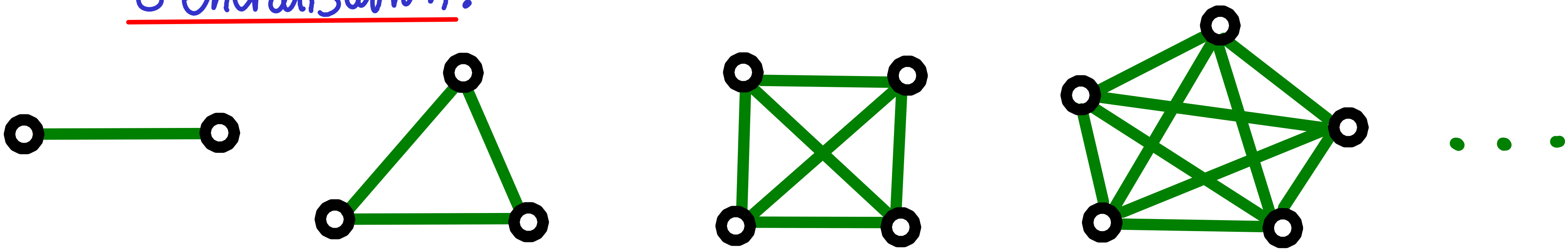

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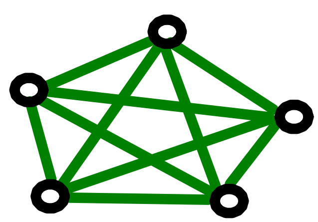
Phase space $\quad \mathcal{M} = \{(P,Q)\} = T^*\mathrm{Hom}(W_0, W_\infty) = \mathrm{Rep}\left(\circ\!\!-\!\!\!-\!\!\!-\!\!\circ, W\right)$

$W = W_0 \oplus W_\infty$

$\big\downarrow$

$\mathcal{M} = \mathrm{Rep}\Big($  $, W\Big)$, $\quad W = \bigoplus_{j \in J} W_j$

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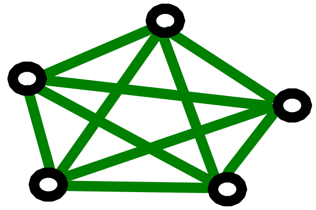
$\mathcal{M} = \mathrm{Rep}\Big($  $, W\Big)$, $\quad W = \bigoplus_{j \in J} W_j$

Point of $\mathcal{M}$ consists of maps $B_{ij} : W_j \to W_i \quad \forall\, i \neq j \in J$

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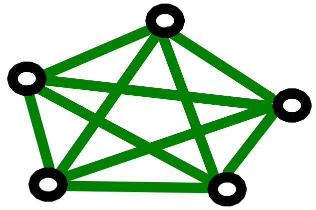
$\mathcal{M} = \mathrm{Rep}\Big($  $, W\Big)$, $W = \bigoplus_{j \in J} W_j$

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<u>Thm</u> • Have (integrable) isomonodromy system

for $\Gamma = \{B_{ij}\}$ w.r.t $\underset{\sim}{T} = \{T_j\}$

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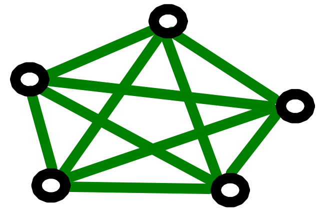
$\mathcal{M} = \mathrm{Rep}\Big($  $, W\Big)$, $W = \bigoplus_{j \in J} W_j$

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<u>Thm</u> • Have (integrable) isomonodromy system for $\Gamma = \{B_{ij}\}$ w.r.t $\underset{\sim}{T} = \{T_j\}$

• Governs isomonodromic deformations of linear differential systems on $\left(\bigoplus_{j \neq \infty} W_j\right) \times \mathbb{P}^1 \to \mathbb{P}^1$

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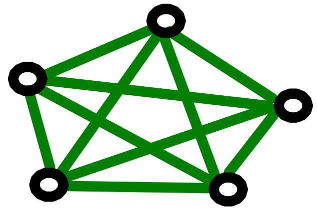
$\mathcal{M} = \mathrm{Rep}\Big($  $, W\Big)$, $W = \bigoplus_{j \in J} W_j$

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• Governs isomonodromic deformations of linear differential systems on $\left(\bigoplus_{j \neq \infty} W_j\right) \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$

• Can act by Möbius transforms on $J \subset \mathbb{P}^1$ to get equiv. system

$J = \{a_j\} \hookrightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, times $T_j \in \mathrm{End}(W_j)$ (diagonalisable)

$$B_{ij} : W_j \to W_i \quad \forall i \neq j \in J$$

Simply-laced isomonodromy system:

$$dB_{ij} = \sum_{k \in J} \widetilde{X_{ik} B_{ki}} B_{ij} + B_{ij} \widetilde{B_{jk} X_{kj}}$$

$$+ \, dT_i \, X_{ik} B_{kj} + B_{ik} X_{kj} \, dT_j - X_{ik} \, dT_k \, X_{kj} / \phi_{ij}$$

$$+ \text{ linear terms}$$

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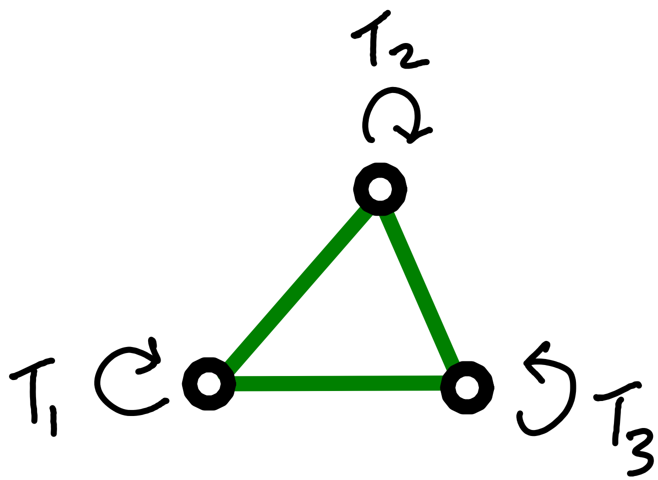
$$+ \text{linear terms}$$

where

$$\phi_{ij} = \begin{cases} (a_i - a_j)^{-1} & \text{if } i, j \neq \infty \\ 1, -1 & j = \infty, i = \infty \text{ resp.} \end{cases}$$

$$X_{ij} = \phi_{ij} B_{ij}, \qquad (B_{ii} = 0)$$

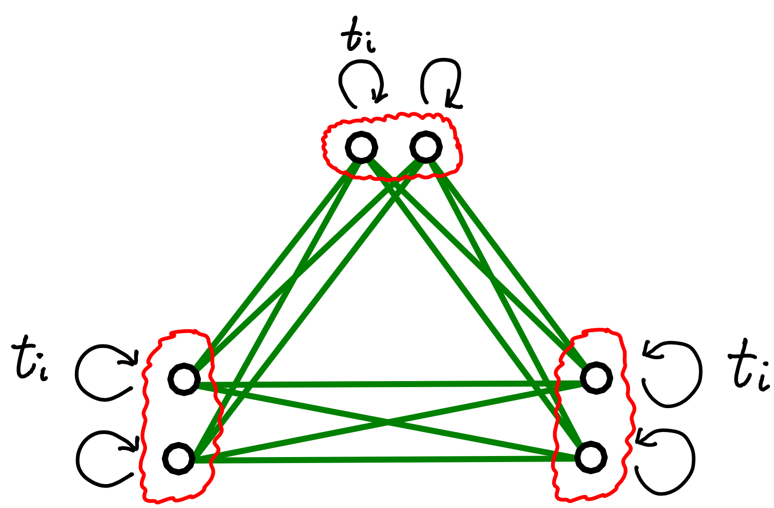
Splay/fission as before:



$I_j = \text{Eigenspaces}(T_j)$

$\Downarrow$

e.g. $|I_j| = 2 \quad \forall j$:

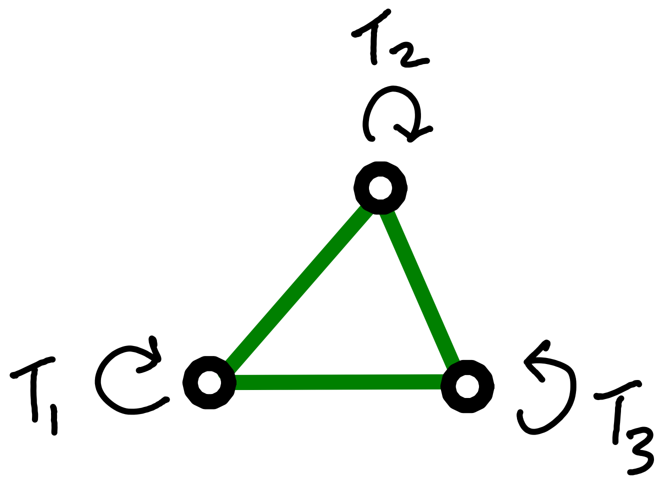


nodes

$I = \bigsqcup_{j \in J} I_j$

$\bigoplus_{i \in I_j} V_i = W_j$

Splay/fission as before:



$I_j = \text{Eigenspaces}(T_j)$

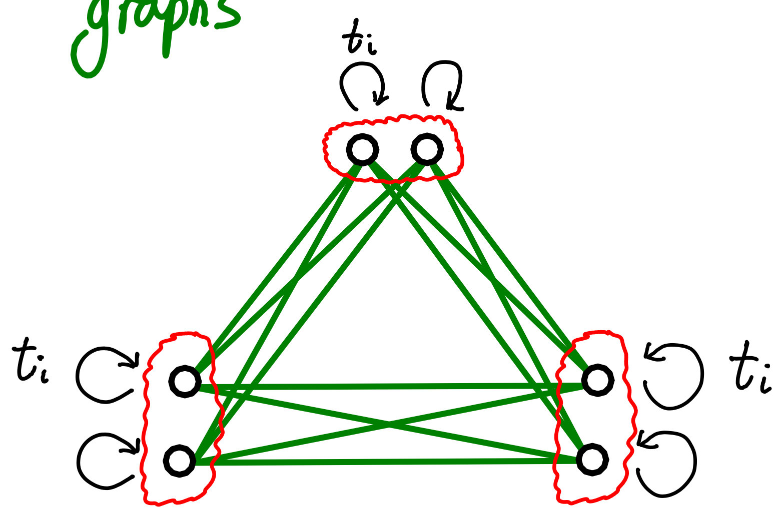
$\Downarrow$

Get all complete k-partite graphs

$k = |J| = \#\text{nodes}$

e.g $|I_j| = 2 \;\; \forall j$:

$\mathcal{G}(2,2,2) \cong$

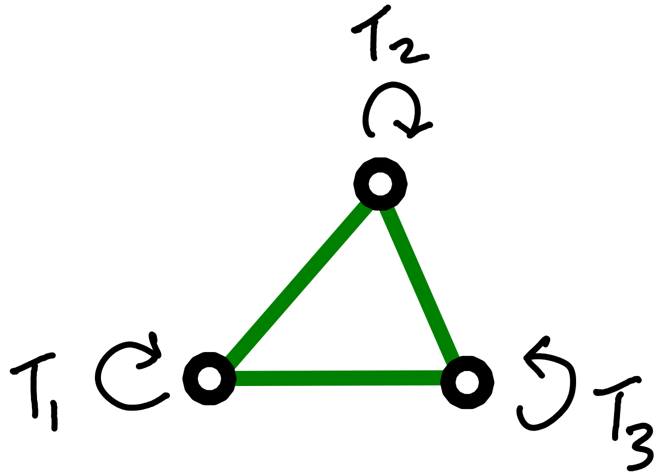


nodes

$I = \bigsqcup_{j \in J} I_j$

$\bigoplus_{i \in I_j} V_i = W_j$

Splay/fission as before:



$I_j = \mathrm{Eigenspaces}(T_j)$

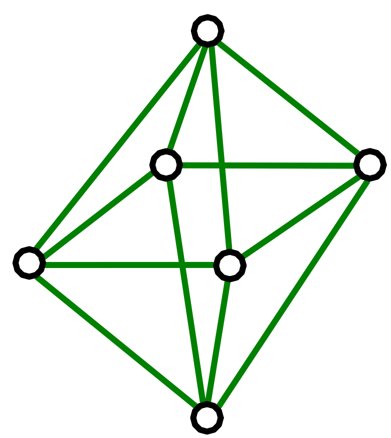
$\Downarrow$

Get all complete k-partite graphs

$k = |J| = \#\text{nodes}$

e.g $|I_j| = 2 \quad \forall j$:

$\mathcal{G}(2,2,2) \cong$

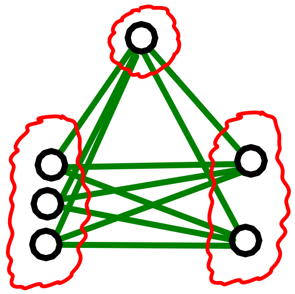


nodes

$$I = \coprod_{j \in J} I_j$$

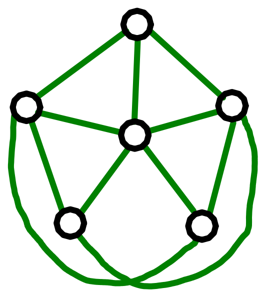
$$\bigoplus_{i \in I_j} V_i = W_j$$

Complete k partite graphs $\Longleftrightarrow$ Integer partitions with k parts



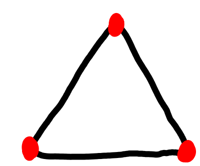
$\Longleftrightarrow$ $1 + 2 + 3 = 6$

Complete k partite graphs $\iff$ Integer partitions with k parts

 $\iff$ $1+2+3 = 6$

E.g. Observe : 1 If $k=|J|=3$, $\dim W_0 = \dim W_0 = \dim W_\infty = 1$

then s. laced IM system $\iff$ Painlevé IV

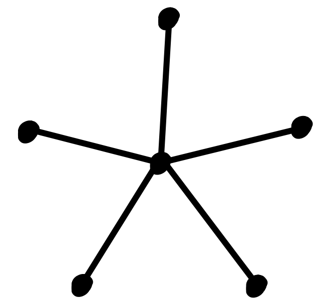
2 (Okamoto '85) Painlevé IV has $A_2^{(1)}$ symmetry 

3 $\mathcal{G}(1,1,1)$ is a triangle

Möbius$(SL_2(\mathbb{C}))$ symmetries (+ Schles. trfms) $\Rightarrow$ Okamoto syms

Graphs from partitions of $N \leq 6$

(omitting totally disconnected graphs $\mathcal{G}(n)$, and stars $\mathcal{G}(n,1)$)



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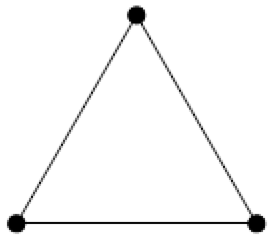

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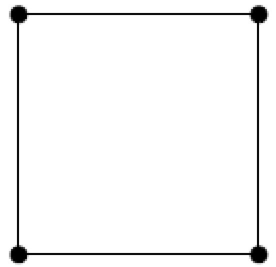

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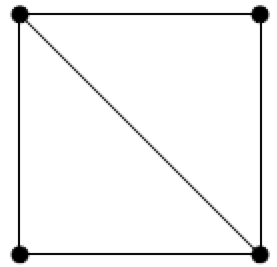

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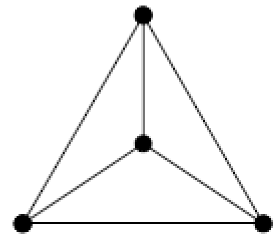

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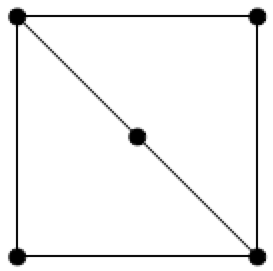

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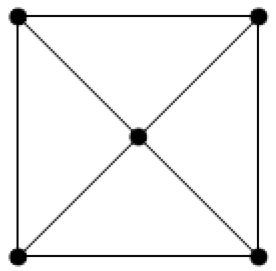

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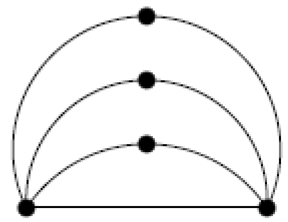

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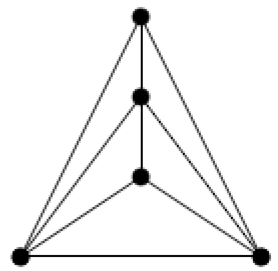

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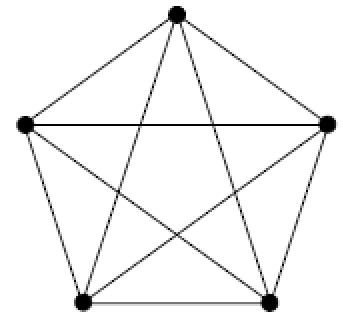

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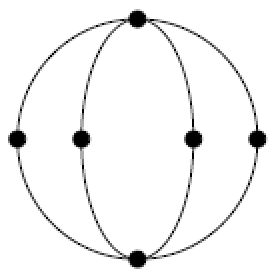

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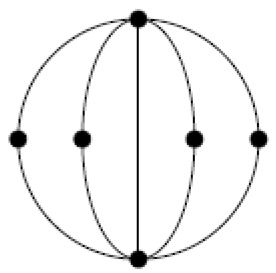

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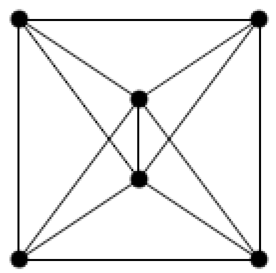

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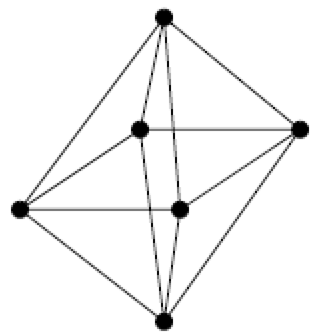

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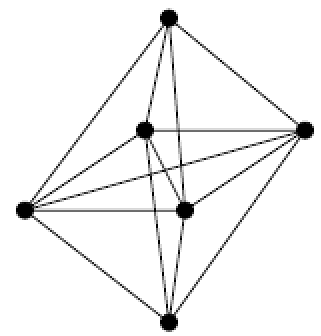

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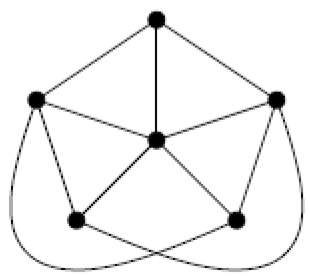

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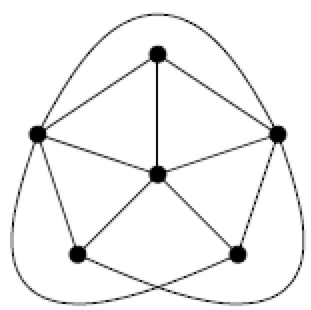

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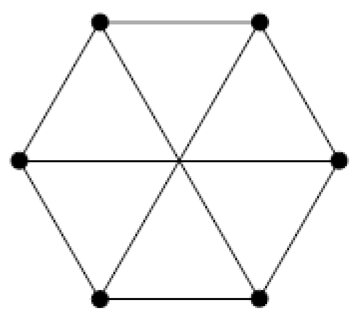

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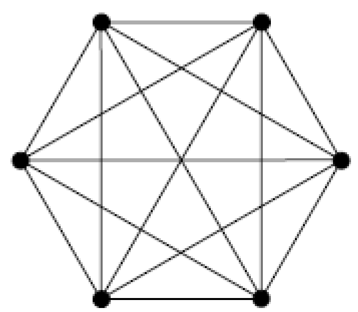

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Ref.s
- arXiv: 0806.1050 Irregular conn.s + KM root systems
- Pub.Math. IHES 2012 Simply-laced isomonodromy systems

Further steps

- Main idea — presentations of modules for Weyl algebra $\mathcal{A}_1$
- Reductions; reduced phase space $\mathcal{M}^*$ $\cong$ Nakajima quiver var.
  $\cong$ moduli of connections
  $\cong$ moduli of $\mathcal{A}_1$-module pres.ns
- Weyl group action via reordering eigenvalues of residues
- Hamiltonians and τ-functions
- Examples: Higher Painlevé systems

Main idea

Let $\mathcal{A}_1 = \mathbb{C}\langle z, \partial \rangle$, $\partial = d/dz$

Suppose α, β, γ $n \times n$ matrices / $\mathbb{C}$

Let $M = \alpha \partial + \beta z - \gamma$

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Consider $\mathcal{A}_1$ modules $\mathcal{N}$ of form

$$\mathcal{A}_1^n \xrightarrow{(\cdot M)} \mathcal{A}_1^n \longrightarrow \mathcal{N} \longrightarrow 0$$

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$$\mathcal{A}_1^n \xrightarrow{(\cdot M)} \mathcal{A}_1^n \longrightarrow \mathcal{N} \longrightarrow 0$$

Lemma This class of modules is stable under the $SL_2(\mathbb{C})$ (symplectic) symmetries of $\mathcal{A}_1$

Everything follows from this (JMMS case $\sim \ker(\alpha) \oplus \ker(\beta) = V = \mathbb{C}^n$)

$V = \mathbb{C}^n$ decomposes into joint eigenspaces of α, β

Eigenvalues $\alpha_i, \beta_i \implies$ point $a_i = -\beta_i/\alpha_i \in \mathbb{P}^1 = \mathbb{C} \cup \infty$

$$V = \bigoplus_{a \in \mathbb{P}^1} W_a = \bigoplus_{j \in J} W_j \qquad (J = \{a_i \text{ occurring}\})$$

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$$\gamma = \underset{\substack{\text{\textbackslash}\\ \mathrm{End}(W_i)}}{(T_i)} + \underset{\substack{\text{\textbackslash}\\ \mathrm{Hom}(W_j, W_i)\ i \neq j}}{(B_{ij})} \in \mathrm{End}\left(\bigoplus W_j\right)$$

(assume T_i semisimple)

$$= \begin{pmatrix} C & 0 \\ 0 & T \end{pmatrix} + \begin{pmatrix} 0 & P \\ Q & B \end{pmatrix} \in \mathrm{End}(W_\infty \oplus U_\infty)$$

$$U_\infty = \bigoplus_{j \neq \infty} W_j, \qquad C = T_\infty$$

$$\gamma = \begin{pmatrix} c & 0 \\ 0 & T \end{pmatrix} + \begin{pmatrix} 0 & P \\ Q & B \end{pmatrix} \in \mathrm{End}\,(W_\infty \oplus U_\infty)$$

$$U_\infty = \bigoplus_{j \neq \infty} W_j$$

Solutions of $\mathcal{N}$ $\Leftrightarrow$ solutions of $\mathrm{End}(U_\infty)$ system:

$$\partial v = \left(Az + B + T + Q(z-c)^{-1}P\right) v$$

$$A = \sum_{j \neq \infty} a_j \, \mathrm{Id}_{W_j}$$

$$\gamma = \begin{pmatrix} c & 0 \\ 0 & T \end{pmatrix} + \begin{pmatrix} 0 & P \\ Q & B \end{pmatrix} \in \mathrm{End}\left(W_\infty \oplus U_\infty\right)$$

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$$\partial v = \Big(\underbrace{Az}_{\text{new term}} + B + T + Q(z-c)^{-1}P\Big)\, v$$

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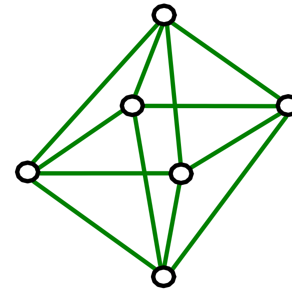
- $SL_2(\mathbb{C})$ action acts by Möbius trfmns on $\{a_j\} \subset \mathbb{P}^1$
- $U_\infty = \bigoplus_{a_j \neq \infty} W_j$ (so rank changes $\Rightarrow k+1$ different values in general) $k = |J|$

Reduction

$\mathbb{M} = \{ B_{\bullet\bullet} \} \cong \{ \rho_{ij} \} = \mathrm{Rep}(\mathcal{G}, V)$ $\Big(\mathcal{G} =$  say $\Big)$

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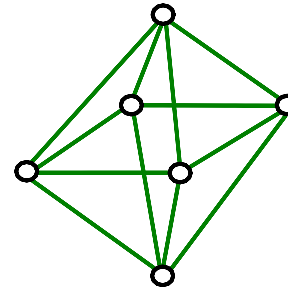
$\mathcal{M} = \{ B_{\bullet\bullet} \} \cong \{ \rho_{ij} \} = \mathrm{Rep}(\mathcal{G}, V)$ $\left(\mathcal{G} = \right.$  say $\left. \right)$

Have action of reductive group $\hat{H} = \prod_{i \in I} GL(V_i)$ preserving SLIM equations

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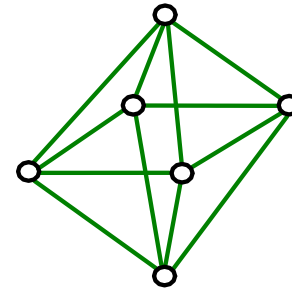
$\mathcal{M} = \{B_{\bullet\bullet}\} \cong \{\rho_{ij}\} = \mathrm{Rep}(\mathcal{G}, V)$ $\Big($ $\mathcal{G} =$  say $\Big)$

Have action of reductive group $\hat{H} = \prod_{i\in I} GL(V_i)$ preserving SLIM equations

Choose (co)adjoint orbit $\check{\mathcal{O}} \subset \mathrm{Lie}(\hat{H}) = \prod \mathfrak{gl}(V_i)$

(i.e. $\check{\mathcal{O}}_i \subset \mathfrak{gl}(V_i) \quad \forall\, i \in I$)

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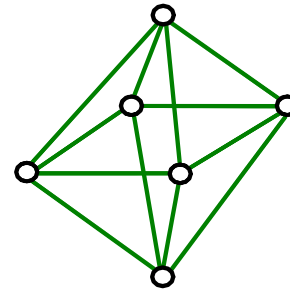
$\mathcal{M} = \{B_{\cdot\cdot}\} \cong \{\rho_{ij}\} = \mathrm{Rep}(\mathcal{G}, V)$ ($\mathcal{G} =$  say)

Have action of reductive group $\hat{H} = \prod_{i \in I} GL(V_i)$ preserving SLIM equations

Choose (co)adjoint orbit $\check{\theta} \subset \mathrm{Lie}(\hat{H}) = \prod \mathfrak{gl}(V_i)$

(i.e. $\check{\theta}_i \subset \mathfrak{gl}(V_i)$ $\forall i \in I$)

Let $\mathcal{M}^* = \mathcal{M} /\!/_{\check{\theta}} \hat{H}$ (symplectic quotient)

— Reduced phase space (really look at stable points)

$\rightsquigarrow \mathcal{M}^* \cong$ a Nakajima quiver var.

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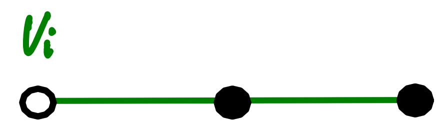
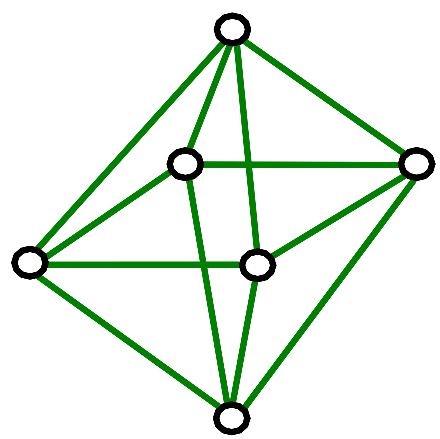
It is known that $\check{\mathcal{O}}_i \subset GL(V_i)$ is a quiver variety

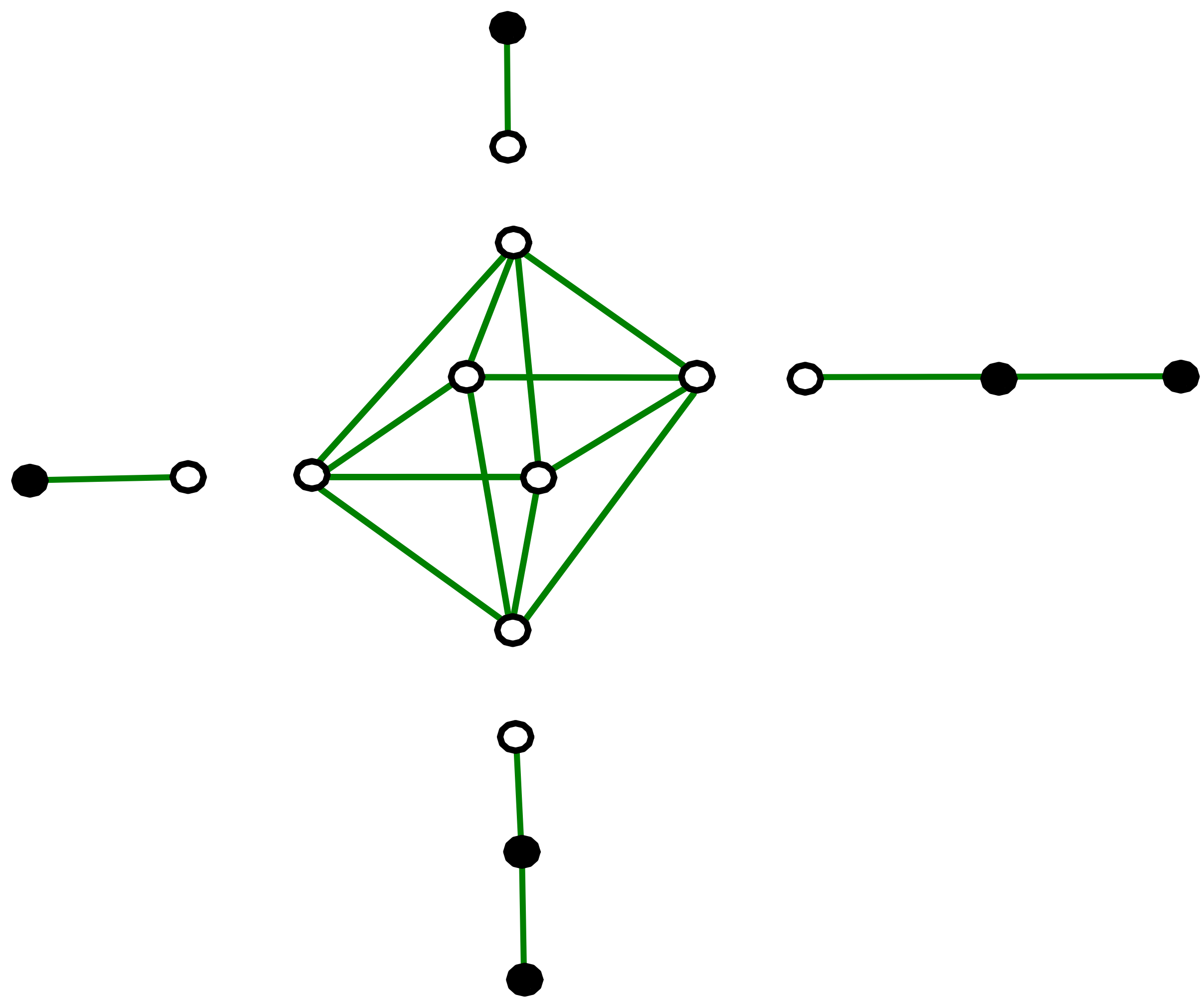
$\check{\mathcal{O}}_i \longleftrightarrow$ 

(Kraft-Procesi, ...)

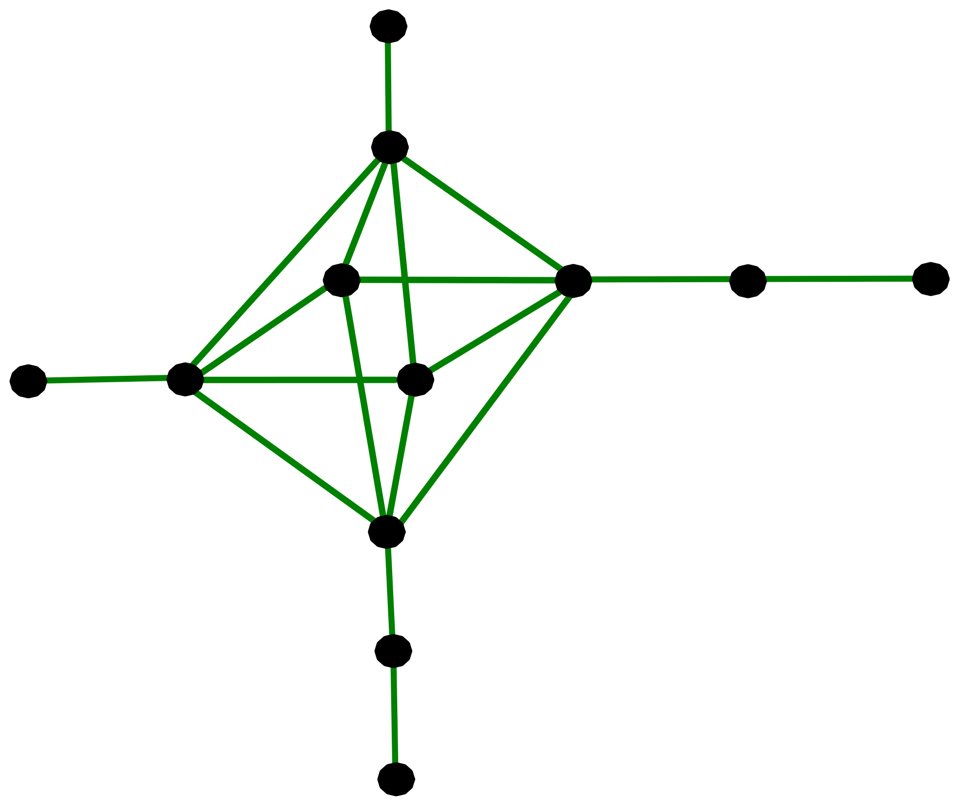
- depends on choice of order of roots of minimal polynomial (of elements of $\check{\mathcal{O}}_i$)

- glue such "legs" on to $\mathcal{G}$

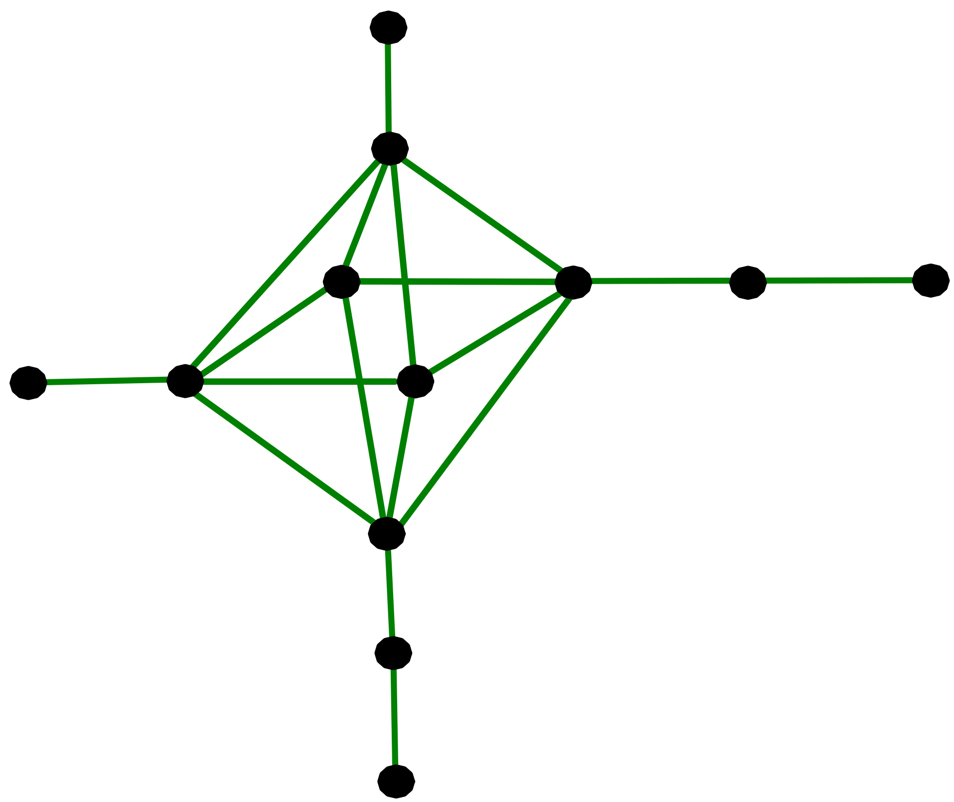
$\mathcal{G} =$ 



$$\hat{\mathcal{G}} =$$



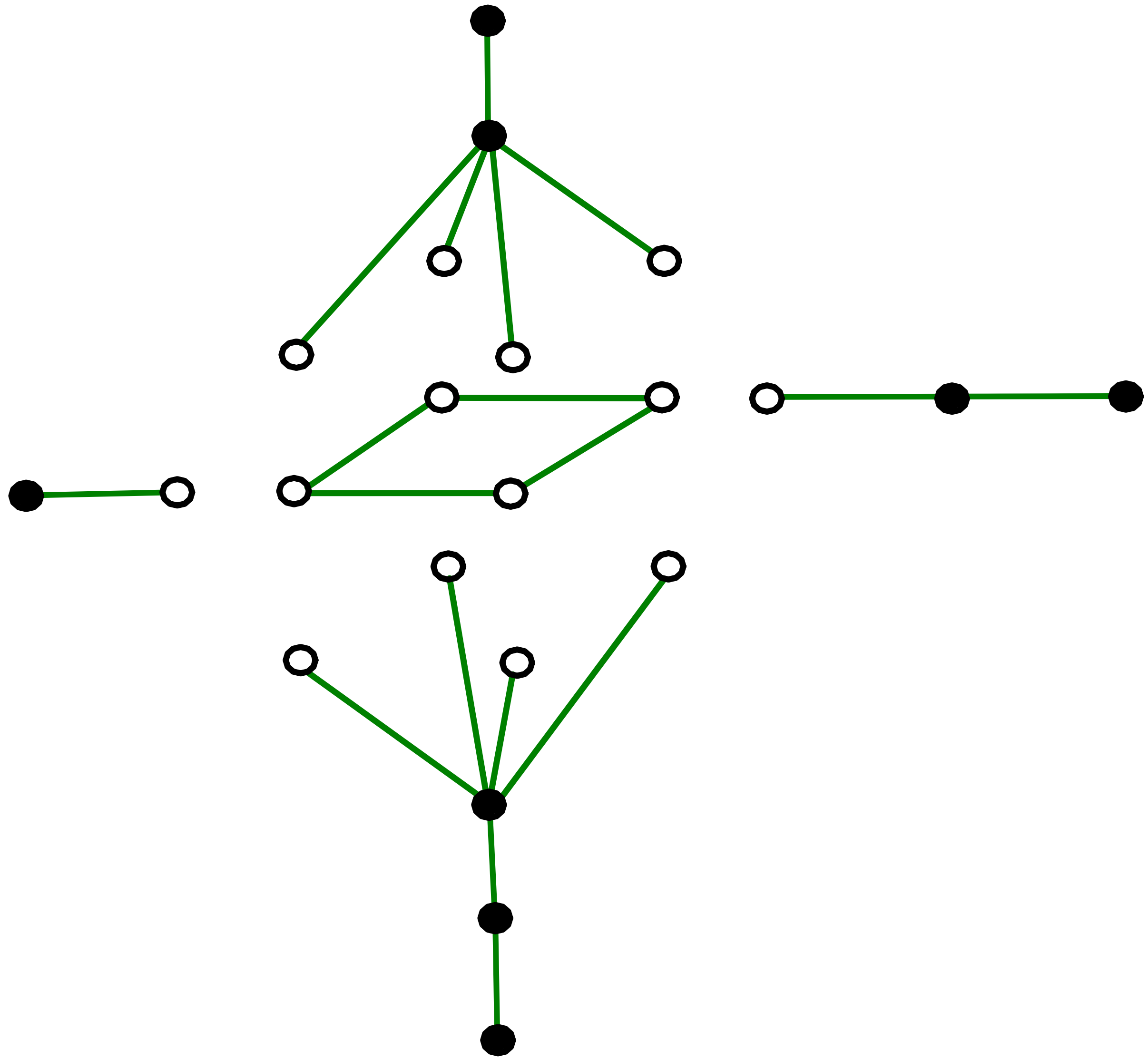
$$\mathcal{M}^* \cong \text{QuiverVar.}(\hat{\mathcal{G}})$$

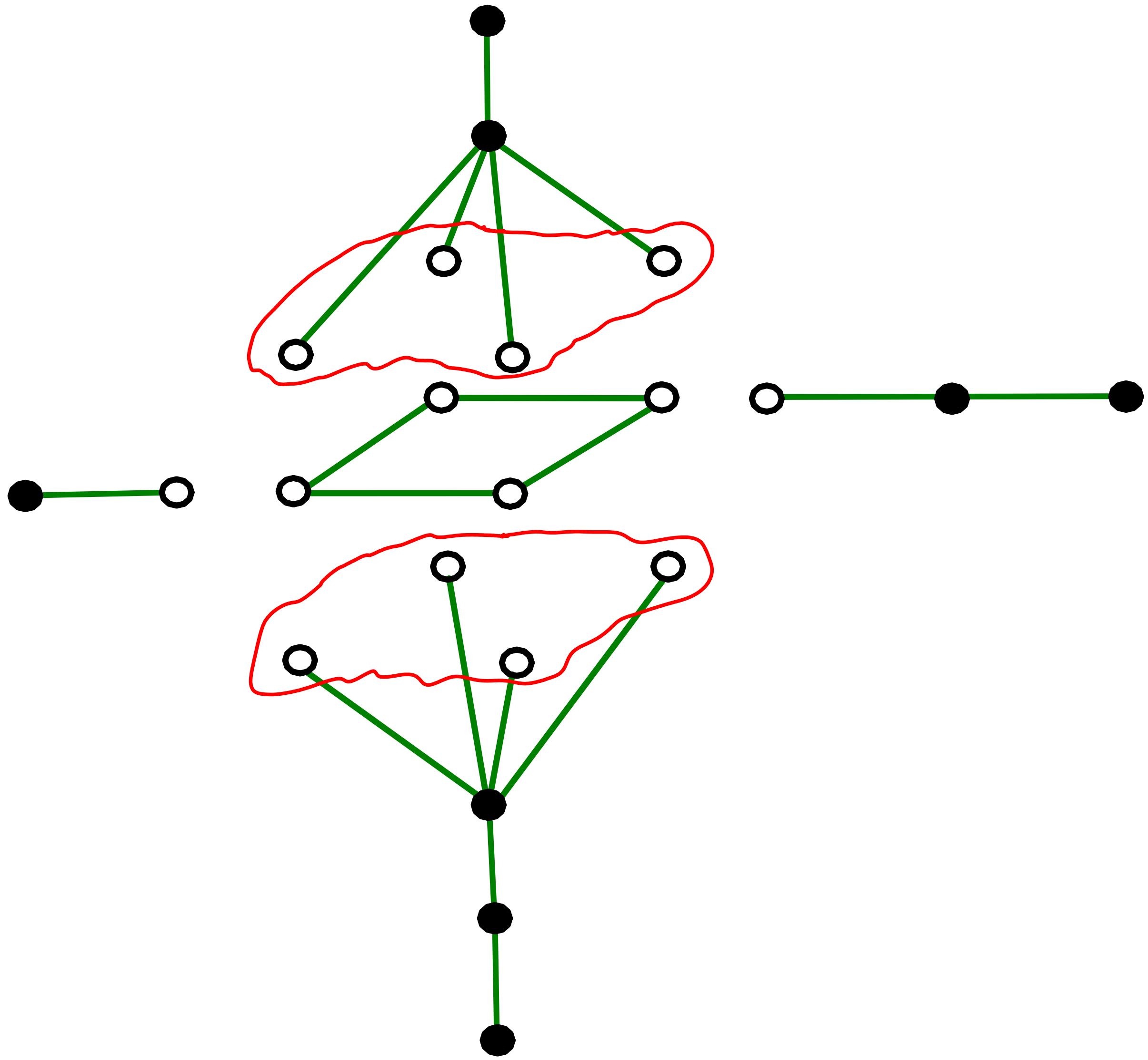
$\hat{\mathcal{G}} =$ 

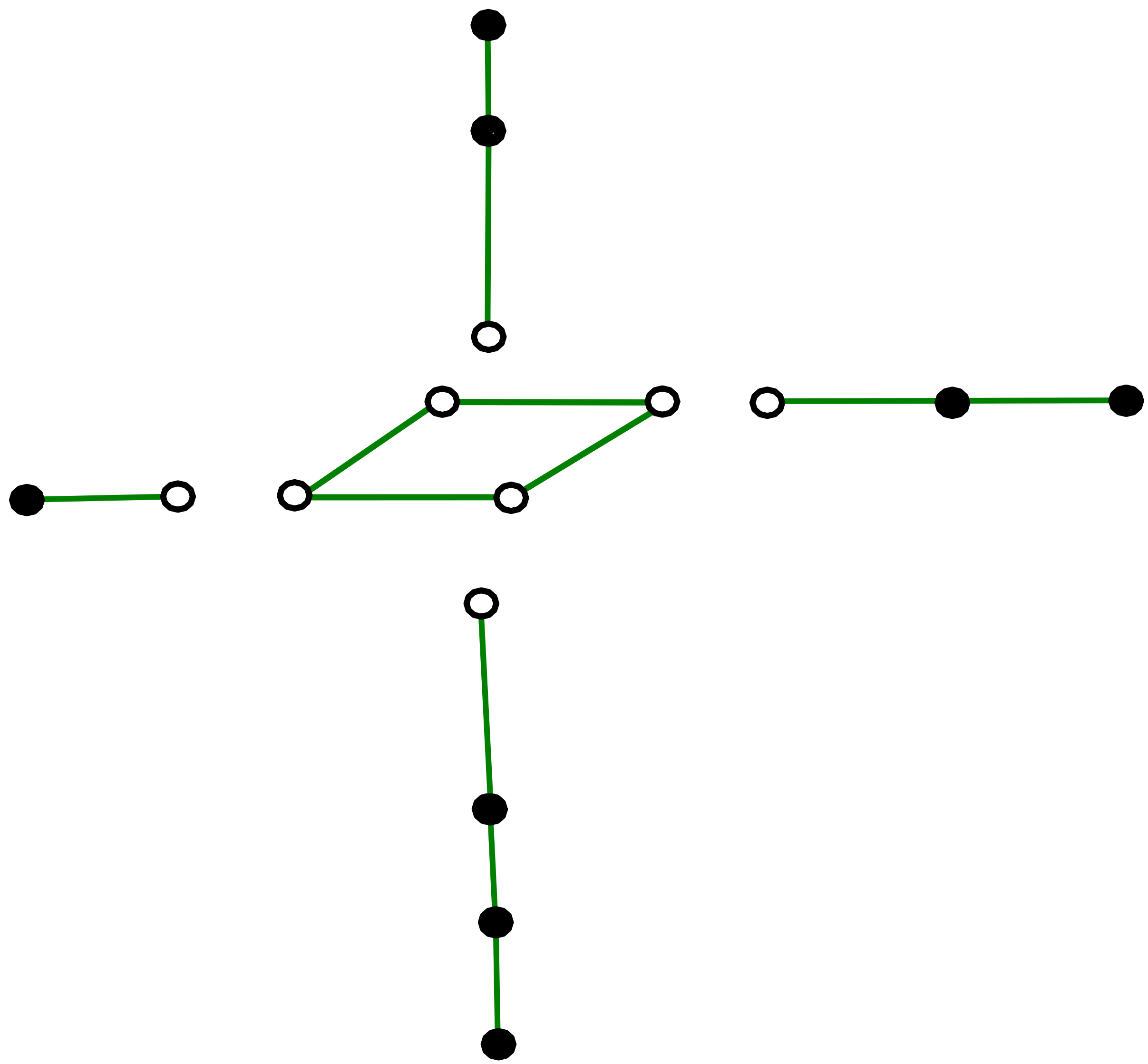
$\mathcal{M}^* \cong \mathrm{QuiverVar.}(\hat{\mathcal{G}})$

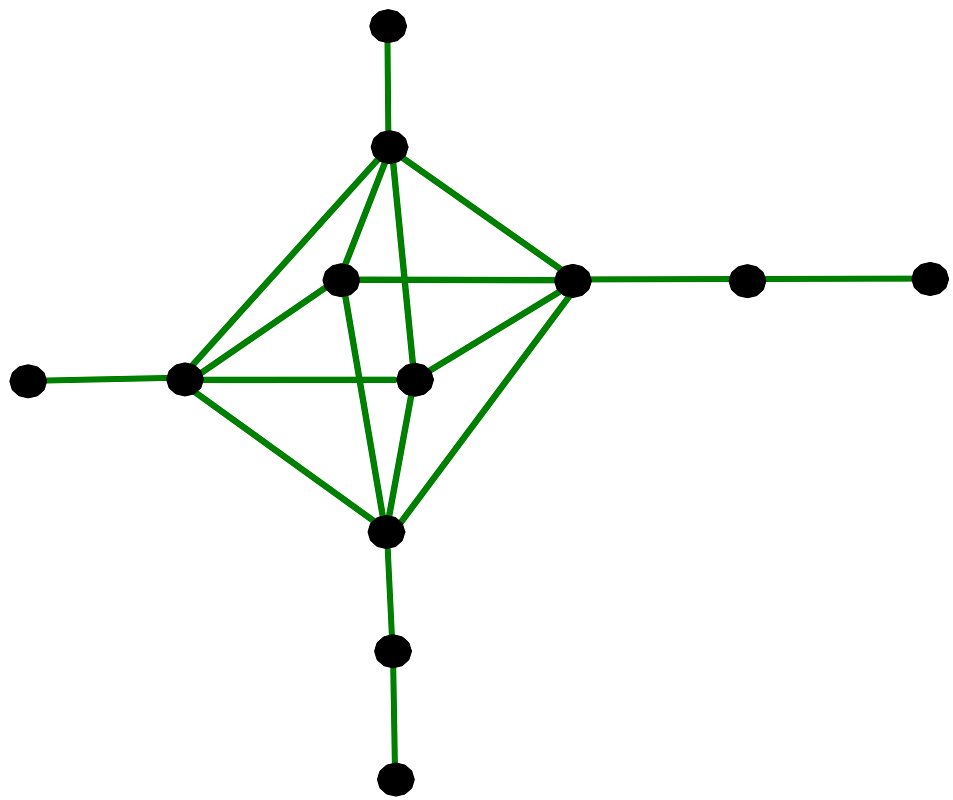
Class of graphs that appear = "Supernova graphs" (complete k-partite + legs)

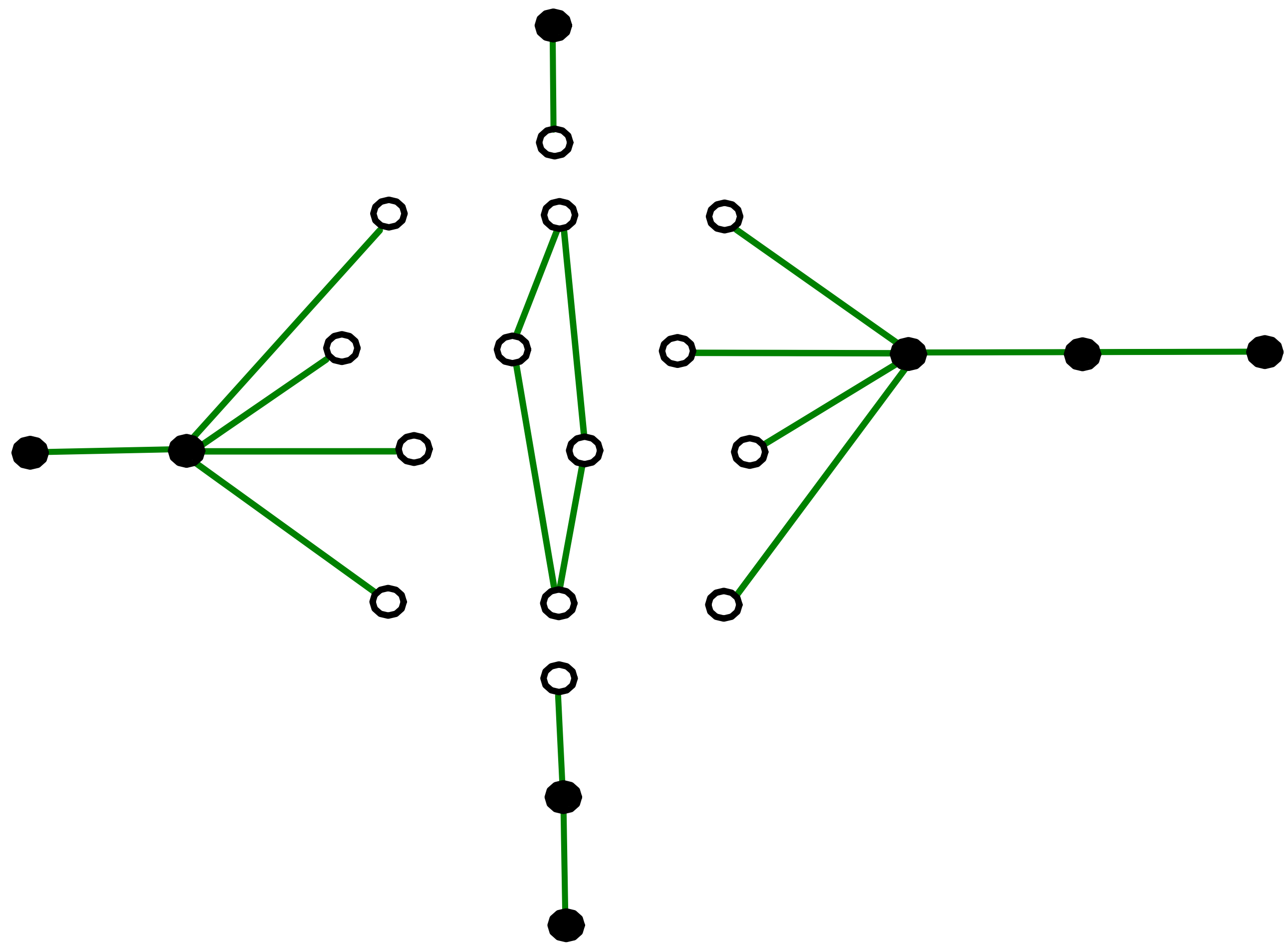
– can attach isomonodromy system to any such graph & its Weyl group gives isomorphisms

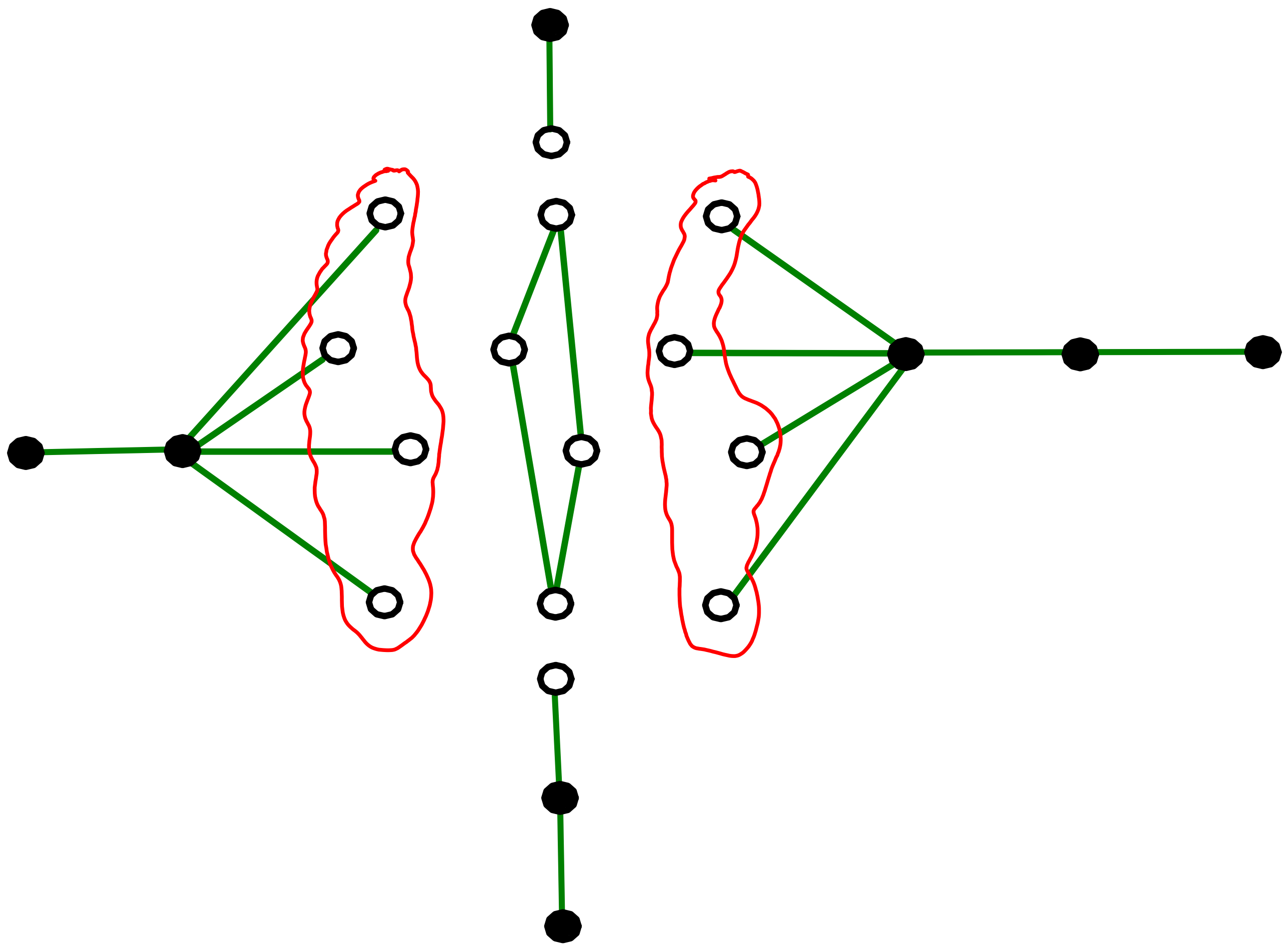


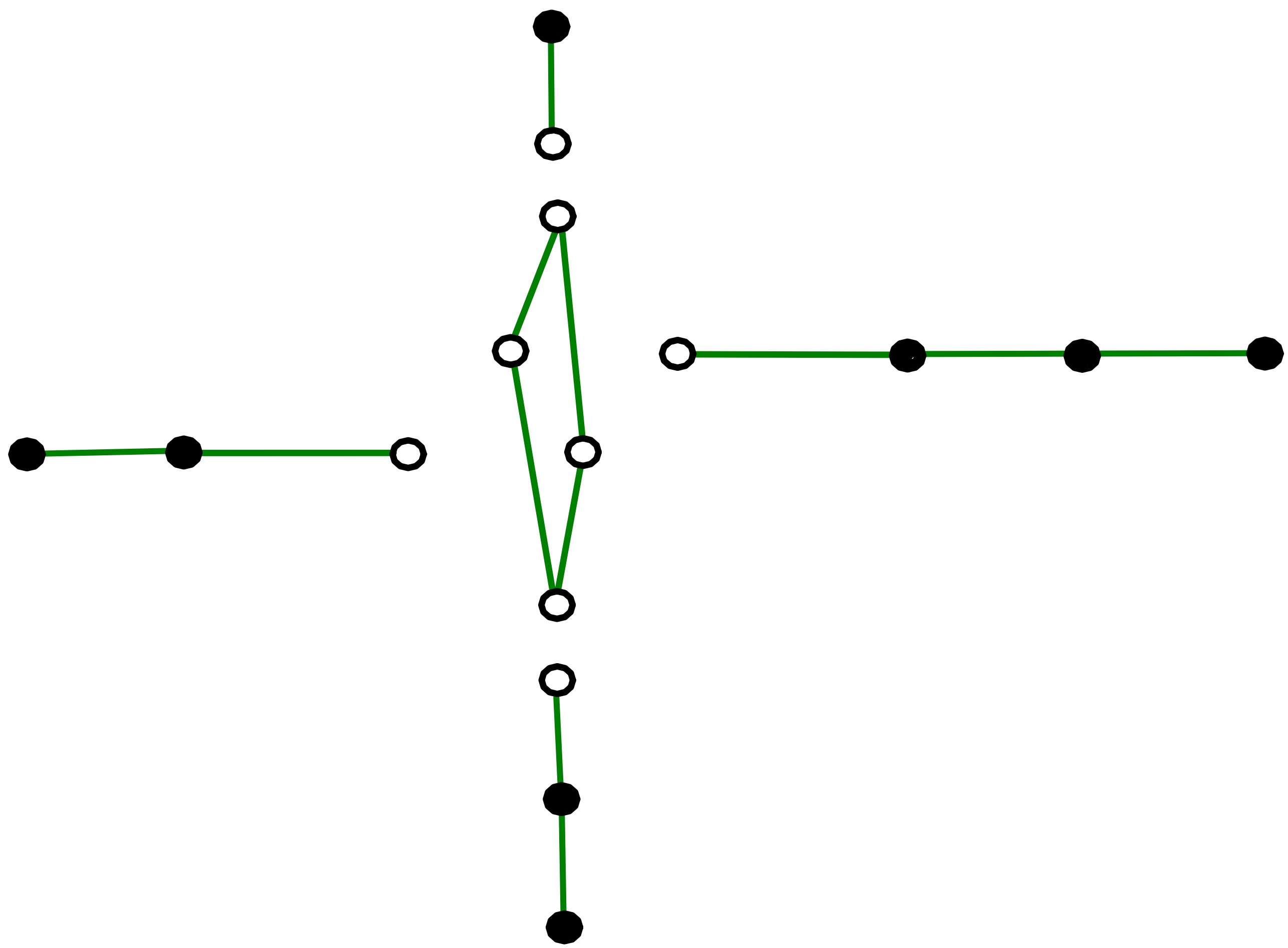


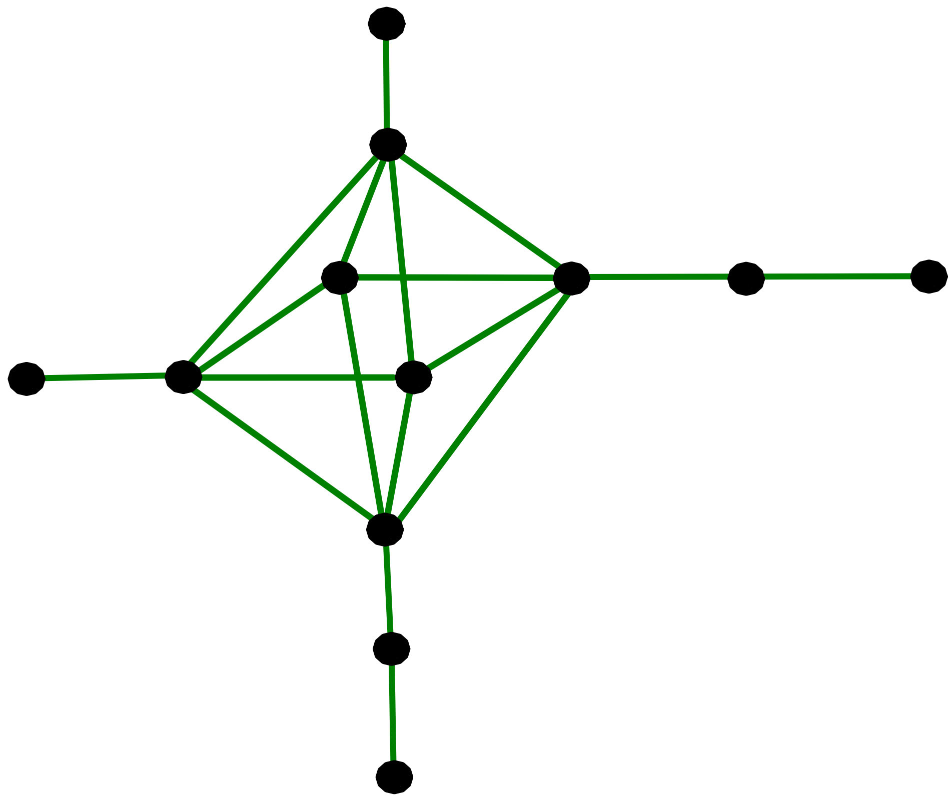












Examples: Higher Painlevé systems

"For any of the (classical, 2nd order) Painlevé equations $X = I, II, \dots, VI$ there is an isomonodromy system hP_X^n of order $2n$ $\forall n = 1, 2, \dots$"

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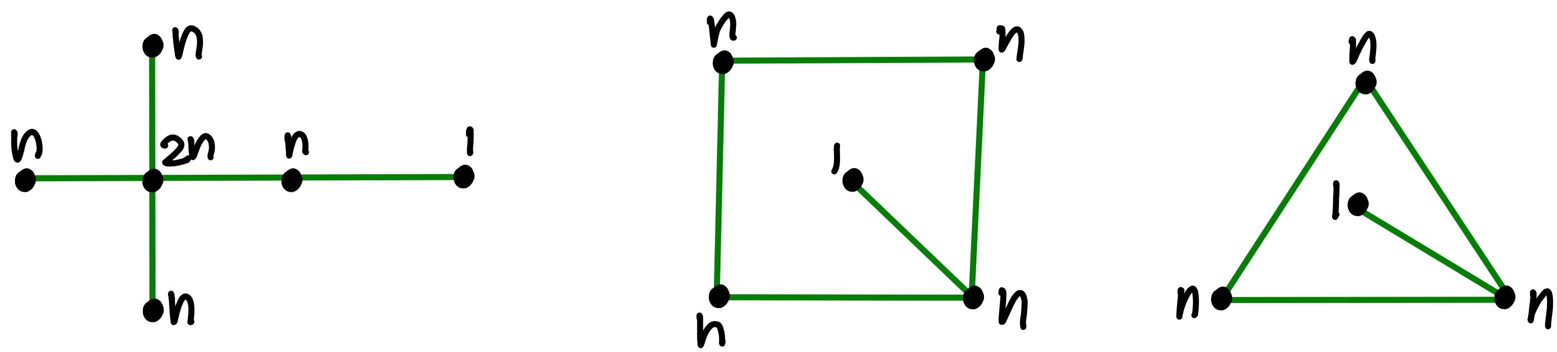
Supernova graph $\hat{\mathcal{G}}$ + vector of dimensions $\Rightarrow$ IMD system



hP_{VI}^n $\qquad$ hP_{V}^n $\qquad$ hP_{IV}^n

complex dimension two, which are related to affine algebras. For example finding spaces of stable connections of complex dimension 4 then (after Theorem 3 below) basically amounts to finding integral vectors of norm -2. (There are infinitely many hyperbolic diagrams that arise in the context of the present paper cf. [33]; five corresponding to the graphs $\Gamma(1111), \Gamma(211), \Gamma(32)$ and the two graphs obtained by attaching a single leg of length one to the square or the triangle, plus 5 star-shaped diagram, 5 with double bonds—see the appendix—and an infinite family with just two nodes and a single higher order edge.) For example one may always take an affine ADE Dynkin diagram with dimension vector the minimal imaginary root δ, then double δ and glue a single leg of length one (with dimension one at the foot) on to the extending node, to obtain a diagram with a dimension vector for a quiver variety of dimension 4. There are other examples however, see Figure 4.

Irreg. Con^{ns} & KM root systems
0806.1050 (June 2008)
(p.12)

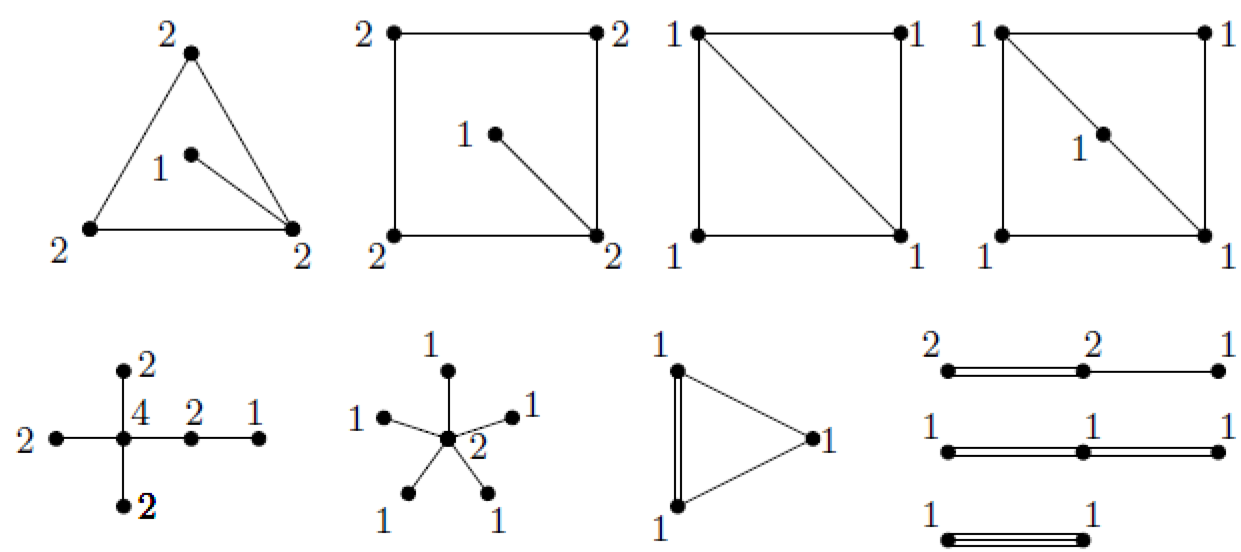


FIGURE 4. Some four dimensional cases.

Link to quiver varieties + work of Nakajima

$$\mathcal{M}^*(hP_X^n) \cong \mathrm{Hilb}^n\big(\mathcal{M}^*(P_X)\big) \quad (X \neq \mathrm{III})$$

Hilbert scheme of n-points

Link to quiver varieties + work of Nakajima $\rightsquigarrow$

$$\mathcal{M}^*(hP_X^n) \cong \mathrm{Hilb}^n(\mathcal{M}^*(P_X)) \quad (X \neq \mathrm{III})$$

Hilbert scheme of n-points

Conjecture Same for full moduli spaces

$$\mathcal{M}(hP_X^n) \cong \mathrm{Hilb}^n(\mathcal{M}(P_X)) \quad (\forall X)$$

(and/or for Higgs bundle version)

Link to quiver varieties + work of Nakajima $\rightsquigarrow$

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Hilbert scheme of n-points

<u>Conjecture</u> Same for full moduli spaces

$$\mathcal{M}(hP_X^n) \cong \mathrm{Hilb}^n(\mathcal{M}(P_X)) \qquad (\forall X)$$

(and/or for Higgs bundle version)

This has been proved recently for tame cases (D_4, E_6, E_7, E_8) by Groechenig

Link to quiver varieties + work of Nakajima $\rightsquigarrow$

$$\mathcal{M}^*(hP_x^n) \cong \mathrm{Hilb}^n(\mathcal{M}^*(P_x)) \qquad (X \neq \mathrm{III})$$

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so "h" can stand for Higher, Hyperbolic or Hilbert