Higgs bundles, connections and quivers

P. Boalch

§1 Quiver varieties

Graph (& data on graph) $\Longrightarrow$ Hyperkahler manifold

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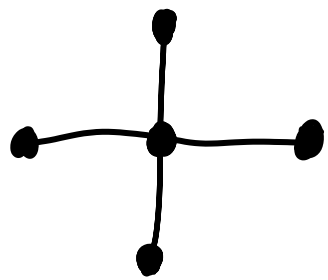
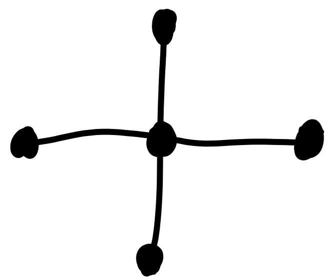


$$\Longrightarrow \quad \mathcal{N}(\Gamma, \underset{\sim}{d}, \underset{\sim}{\lambda})$$

§1 Quiver varieties

Graph (& data on graph) $\Rightarrow$ Hyperkahler manifold



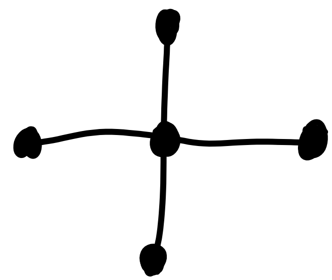
$$\Rightarrow \quad \mathcal{N}(\Gamma, \underset{\sim}{d}, \underset{\sim}{1})$$

§2 Wild Hitchin moduli spaces

Curve (and data on the curve) $\Rightarrow$ Hyperkahler manifold

§1 Quiver varieties

Graph (& data on graph) $\Longrightarrow$ Hyperkahler manifold



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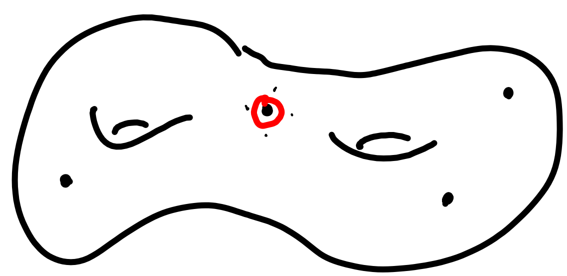

$$\Longrightarrow \quad \mathcal{M}(\Sigma, e)$$

§1 Quiver varieties

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§3 Compare / Relate these stories

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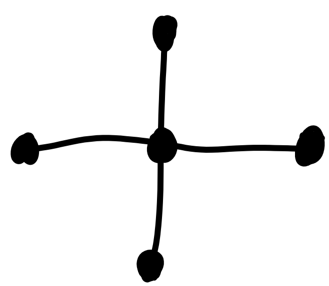

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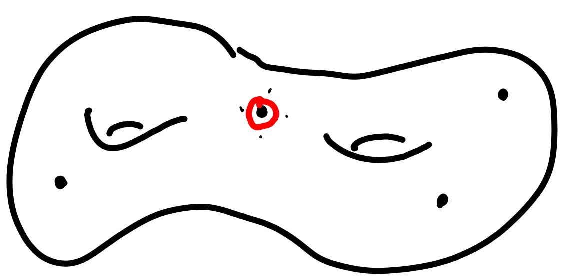

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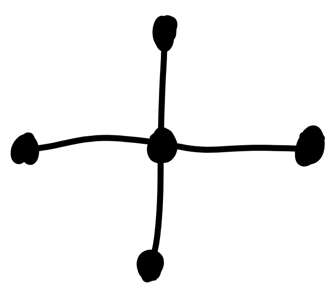

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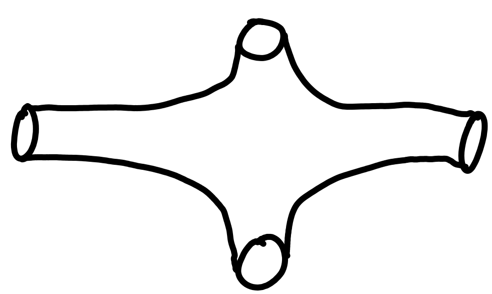

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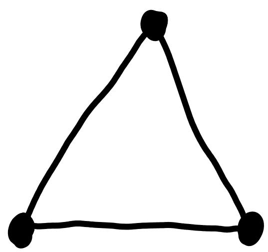 $\Rightarrow \quad \mathcal{N}(\Gamma, \underset{\sim}{d}, \underset{\sim}{1})$

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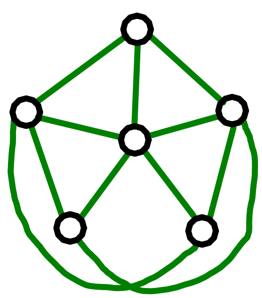 $\Rightarrow$ $\mathcal{N}(\Gamma, \underset{\sim}{d}, \underset{\sim}{1})$

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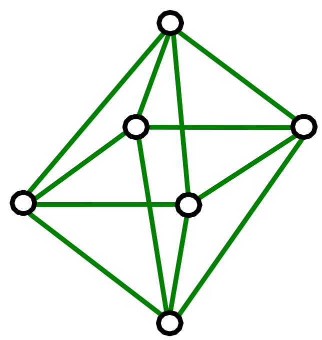

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?

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§1 Quiver varieties

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Kronheimer, Nakajima (1990's) attached hyperkähler manifolds to graphs

graph Γ $\Rightarrow$ $\mathcal{N}(\Gamma, \lambda, d)$

ALE spaces, instantons on ALE spaces ....

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Kronheimer, Nakajima (1990's) attached hyperkahler manifolds to graphs

$$\text{graph } \Gamma \Longrightarrow \mathcal{N}(\Gamma, \lambda, d)$$

Γ graph with nodes I, $V = \bigoplus_{i \in I} V_i$ (I graded vector space)

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$$\mathrm{Rep}(\Gamma, V) = \bigoplus_{e \in \Gamma} \mathrm{Hom}(V_{t(e)}, V_{h(e)}) \qquad t(e) \xrightarrow{e} h(e)$$

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$$\mathrm{Rep}(\Gamma, V) = \bigoplus_{e \in \overline{\Gamma}} \mathrm{Hom}(V_{t(e)}, V_{h(e)}) \qquad t(e) \xrightarrow{e} h(e)$$

$$G = \prod_I GL(V_i) \circlearrowright \mathrm{Rep}(\Gamma, V) \quad \& \quad \boxed{\mathcal{N}(\Gamma, \lambda, d) = \mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda}\, G}$$

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$$\Downarrow$$

Kac-Moody algebra (Cartan matrix $C = 2 -$ adjacency matrix)

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$$\mathrm{Dim}_{\mathbb{C}}(\mathcal{N}(\Gamma, \lambda, d)) = 2 - (d, d) = 2 - d \cdot Cd$$

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Kronheimer, Nakajima (1990's) attached hyperkähler manifolds to graphs

graph $\Gamma \Rightarrow \mathcal{N}(\Gamma, \lambda, d) = \mathrm{Rep}(\Gamma, v) /\!/_{\lambda} G$

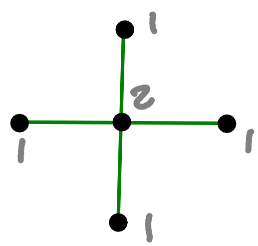
$\Downarrow$

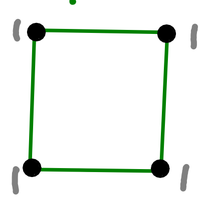
Kac-Moody algebra (Cartan matrix $C = 2 -$ adjacency matrix)

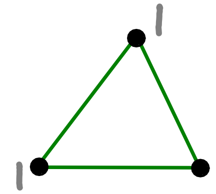
$$\mathrm{Dim}_{\mathbb{C}}(\mathcal{N}(\Gamma, \lambda, d)) = 2 - (d, d) = 2 - d \cdot Cd$$

e.g. Γ affine ADE Dynkin graph, $d =$ min. imaginary root

$(d,d) = 0$, $\mathrm{Dim}_{\mathbb{C}} \mathcal{N} = 2 \Rightarrow$ hyperkähler four manifold (ALE)

e.g. 





$\sim \widehat{\mathbb{C}^2/\mathbb{Z}_3}$

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$\mathrm{Dim}_{\mathbb{C}} = 2 - (d,d)$

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$\mathrm{Dim}_{\mathbb{C}} = 2-(d,d)$

Reflection isomorphisms

If $i \in I$ & $\lambda_i \neq 0$ then $\mathcal{N}(\Gamma,\lambda,d) \cong \mathcal{N}(\Gamma, r_i(\lambda), s_i(d))$

$$\left.\begin{array}{l} r_i \circlearrowright \mathbb{C}^I \\ s_i \circlearrowright \mathbb{Z}^I \end{array}\right\} \text{reflections generating Weyl group } W(\Gamma)$$

§2 Wild Hitchin moduli spaces

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Fix $G = GL_n(\mathbb{C})$

Σ

smooth
compact
curve

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Fix $G = GL_n(\mathbb{C})$

$$\Sigma \Longrightarrow H^1(\Sigma, G)$$

smooth
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§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

$$\Sigma \Longrightarrow H^1(\Sigma, G) \text{ ——— } \{\text{Connections } (V, D)\}/\text{isom.}$$

smooth compact curve

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

$$\Sigma \implies H^1(\Sigma, G) \longleftrightarrow \{\text{connections } (V, \nabla)\} / \text{isom.}$$

smooth compact curve

$V \to \Sigma$ rank n holomorphic vector bundle

$\nabla : V \to V \otimes \Omega^1_\Sigma$ connection

$$\nabla(fs) = (df)s + f\nabla(s)$$

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

Σ smooth compact curve $\Longrightarrow H^1(\Sigma, G)$

$H^1(\Sigma, G)$ — $\{\text{connections } (V, \nabla)\}/\text{isom.}$

$\qquad\qquad\qquad \Big\| \ \text{RH correspondence}$

$H^1(\Sigma, G)$ — $\mathrm{Hom}(\pi_1(\Sigma), G)/G$

$V \to \Sigma$ rank n holomorphic vector bundle

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Fix $G = GL_n(\mathbb{C})$

$$\Sigma \Longrightarrow \left\{\text{Connections } (V, \nabla)\right\} / \text{isom.}$$

smooth compact curve

$$\Big\| \text{ RH correspondence}$$

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G$$

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

$$\Sigma \Longrightarrow \left\{ \text{stable connections } (V, \nabla) \right\} / \text{isom.}$$

smooth compact curve

$$\Big\| \text{ RH correspondence}$$

$$\mathrm{Hom}^{irr}(\pi_1(\Sigma), G)/G$$

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

Σ smooth compact curve $\Longrightarrow$

$$\mathcal{M}_{DR}(\Sigma) = \left\{ \overset{\text{stable}}{\text{connections}} \, (V, \nabla) \right\} / \text{isom.}$$

$\Vert\!\!\wr$ RH isomorphism

$$\mathcal{M}_B(\Sigma) = \mathrm{Hom}^{irr}(\pi_1(\Sigma), G) / G$$

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

$\Sigma \Longrightarrow$ $\mathcal{M}_{DR}(\Sigma)$

smooth compact curve

$\Big\|\!\Big\wr$ RH isomorphism

$\mathcal{M}_B(\Sigma)$

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

{solutions to Hitchin's equations}

$\shortparallel$

$$\Sigma \Longrightarrow \mathcal{M}(\Sigma) \underset{\text{Corlette-Donaldson}}{\approx} \mathcal{M}_{DR}(\Sigma)$$

(Hitchin-Kobayashi for connections)

smooth compact curve

Hyperkahler manifold

$$\mathcal{M}_{DR}(\Sigma) \;\equiv\; \mathcal{M}_B(\Sigma) \quad \text{RH isomorphism}$$

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

$$\mathcal{M}_{Dol}(\Sigma) = \left\{ \text{stable Higgs bundles } (E, \Phi) \right\} / \text{isom.}$$

$$\underset{\text{smooth compact curve}}{\Sigma} \Longrightarrow \underset{\text{Hyperkahler manifold}}{\mathcal{M}(\Sigma)} \underset{\text{Corlette-Donaldson}}{\cong} \mathcal{M}_{DR}(\Sigma)$$

$$\mathcal{M}_{DR}(\Sigma) \;\Big|\Big|\Big|\; \text{RH isomorphism} \;\; \mathcal{M}_B(\Sigma)$$

$E \longrightarrow \Sigma$ rank n holomorphic vector bundle

$\Phi : E \longrightarrow E \otimes \Omega^1_\Sigma$ Higgs field

$\Phi(fs) = \cancel{(df)}s + f\Phi(s)$

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

(Hitchin–Kobayashi for Higgs bundles)

$$\mathcal{M}(\Sigma) \overset{\text{Hitchin–Simpson}}{\equiv} \mathcal{M}_{Dol}(\Sigma)$$

$$\Sigma \Longrightarrow \mathcal{M}(\Sigma) \underset{\text{Corlette–Donaldson}}{\equiv} \mathcal{M}_{DR}(\Sigma)$$

smooth compact curve

Hyperkahler manifold

$$\mathcal{M}_{DR}(\Sigma) \cong \mathcal{M}_B(\Sigma) \quad \text{RH isomorphism}$$

$E \to \Sigma$ rank n holomorphic vector bundle

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$$\Phi(fs) = \cancel{(df)s} + f\Phi(s)$$

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

$$\underset{\text{smooth compact curve}}{\Sigma} \Longrightarrow \underset{\text{Hyperkähler manifold}}{\mathcal{M}(\Sigma)} \begin{array}{l} \overset{\text{Hitchin-Simpson}}{=\!=\!=} \mathcal{M}_{Dol}(\Sigma) \\ \qquad\qquad\quad \Big\| \; \text{Non-abelian Hodge} \\ \underset{\text{Corlette-Donaldson}}{=\!=\!=} \mathcal{M}_{DR}(\Sigma) \\ \qquad\qquad\quad \Big\| \; \text{RH isomorphism} \\ \qquad\qquad\;\; \mathcal{M}_B(\Sigma) \end{array}$$

3 algebraic structures

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

$$\Sigma \Longrightarrow \mathcal{M}(\Sigma)$$

smooth compact curve

Hyperkahler manifold

Hitchin-Simpson $\mathcal{M}_{Dol}(\Sigma)$

$\|\wr$ Non-abelian Hodge

Corlette-Donaldson $\mathcal{M}_{DR}(\Sigma)$

$\|\wr$ RH isomorphism

$\mathcal{M}_{B}(\Sigma)$

3 algebraic structures

Generalisations

1. Σ higher dimensions
   (no new moduli spaces known)

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

$$\Sigma \Longrightarrow \mathcal{M}(\Sigma)$$

smooth compact curve

Hyperkahler manifold

$\mathcal{M}(\Sigma) \cong \mathcal{M}_{Dol}(\Sigma)$ Hitchin-Simpson

$\mathcal{M}_{Dol}(\Sigma)$

$\parallel\wr$ Non-abelian Hodge

$\mathcal{M}(\Sigma) \cong \mathcal{M}_{DR}(\Sigma)$ Corlette-Donaldson

$\mathcal{M}_{DR}(\Sigma)$

$\parallel\wr$ RH isomorphism

$\mathcal{M}_B(\Sigma)$

3 algebraic structures

Generalisations

1. Σ higher dimensions
   (no new moduli spaces known)
2. Σ not compact/meromorphic connections
   (lots of new complete hyperkahler moduli spaces)

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

Σ

irregular curve

("wild Riemann Surface")

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

smooth compact curve

$(a_1, \dots, a_m)$ m distinct points of Σ

$$\Sigma = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$$

irregular curve

("wild Riemann Surface")

$(Q_1, \dots, Q_m)$ $\quad Q_i$: irregular type at a_i

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Fix $G = GL_n(\mathbb{C})$

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$(Q_1, \dots, Q_m)$ Q_i: irregular type at a_i

Def^n Fix $T \subset G$. An "irregular type" at $a \in \Sigma$ is an element

$$Q \in \mathfrak{t}(\hat{K}_a)/\mathfrak{t}(\hat{\mathcal{O}}_a) \qquad (\mathfrak{t} = \mathrm{Lie}(T))$$

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Fix $G = GL_n(\mathbb{C})$

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Def^{n} Fix $T \subset G$. An "irregular type" at $a \in \Sigma$ is an element

$$Q \in \mathfrak{t}(\hat{K}_a)/\mathfrak{t}(\hat{\mathcal{O}}_a) \qquad (\mathfrak{t} = \mathrm{Lie}(T))$$

$$= \mathfrak{t}((z))/\mathfrak{t}[[z]] \quad \text{if} \quad z(a) = 0$$

So $Q = A_r/z^r + \dots + A_1/z$ for some $A_i \in \mathfrak{t}$, $r \geq 0$

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

smooth compact curve

$(a_1, \dots, a_m)$ m distinct points of Σ

$$\boldsymbol{\Sigma} = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$$

irregular curve

("wild Riemann Surface")

$(Q_1, \dots, Q_m)$ Q_i irregular type at a_i

$\left(Q = A_r/z^r + \dots + A_1/z\right)$

[more generally replace $\mathfrak{t}((z)) \subset \mathfrak{g}((z))$

by non-conjugate Cartan $\hat{\mathfrak{t}} \subset \mathfrak{g}((z))$

we'll ignore this 'twisted' case here]

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

$$\Sigma = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q}) \Longrightarrow$$

irregular curve

$$\mathcal{M}_{DR}(\Sigma)$$

$$\Big|\Big|\Big\wr \;\; \text{RH} \quad \text{irregular isomorphism}$$

$$\mathcal{M}_{B}(\Sigma)$$

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

$$\underset{\text{irregular curve}}{\boldsymbol{\Sigma}} = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q}) \Longrightarrow \mathcal{M}_{DR}(\boldsymbol{\Sigma})$$

$$\mathcal{M}_{DR}(\boldsymbol{\Sigma}) \underset{\text{RH}}{\cong} \mathcal{M}_B(\boldsymbol{\Sigma}) \quad \text{irregular RH isomorphism}$$

Locally near a_i:

$$\nabla \cong d - \Big(\underbrace{dQ_i}_{\text{irregular part specified by irregular type}} + \Lambda \frac{dz}{z} + \cdots\Big)$$

WLOG $\Lambda \in \mathfrak{h}_i := C_{\mathfrak{g}}(Q_i)$

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

conjugacy class (& weights)

$\mathcal{C} \subset \underset{\sim}{H} = H_1 \times \cdots H_m$ $\left(H_i = C_G(Q_i)\right)$

$$\Sigma = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q}) \Longrightarrow \mathcal{M}_{DR}(\Sigma, \mathcal{C})$$

irregular curve

$$\mathcal{M}_{DR}(\Sigma, \mathcal{C}) \underset{\text{RH}}{\overset{\text{irregular}}{\cong}} \mathcal{M}_B(\Sigma, \mathcal{C})$$

irregular RH isomorphism

Locally near a_i:

$$\nabla \cong d - \Big(\underbrace{dQ_i}_{\text{irregular part specified by irregular type}} + \Lambda \frac{dz}{z} + \cdots\Big)$$

WLOG $\Lambda \in H_i := C_g(Q_i)$

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

....

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

(weighted) conjugacy class $\mathcal{C} \subset \underset{\sim}{H}$

$\Sigma \Longrightarrow \mathcal{M}(\Sigma, \mathcal{C})$

irregular curve

Hyperkahler manifold

"wild Hitchin space"

(Biquard-B. '04)

Hitchin-Simpson

Biquard-B.

Corlette-Donaldson

Sabbah

$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$

$\Big|\Big|\Big|$ Wild non-abelian Hodge isom.

$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$

$\Big|\Big|\Big|$ irregular RH isomorphism

$\mathcal{M}_{B}(\Sigma, \mathcal{C})$

(See e.g. survey 1203.6607 for full details)

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

njugacy class

$c\ \underset{\sim}{H}$

lar

e

$$\Longrightarrow \quad \mathcal{M}(\Sigma, \mathcal{C})$$

Hyperkahler manifold

"wild Hitchin space"

(Biquard - B. '04)

$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$ — Hitchin-Simpson / Biquard-B.

$\Vert\wr$ Wild non-abelian Hodge isom.

$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$ — Corlette-Donaldson / Sabbah

$\Vert\wr$ irregular RH isomorphism

$\mathcal{M}_{B}(\Sigma, \mathcal{C})$

g. survey 1203.6607 for full details)

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Fix $G = GL_n(\mathbb{C})$

$$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$$

Hitchin-Simpson

Biquard-B.

$\Big|\Big|\Big|\!\!\wr$ Wild non-abelian Hodge isom.

$$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$$

Corlette-Donaldson

Sabbah

$\Big|\Big|\Big|\!\!\wr$ RH irregular isomorphism

$$\mathcal{M}_{B}(\Sigma, \mathcal{C})$$

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§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

$$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$$

$\cong$ wild non-abelian Hodge isom.

$$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$$

$\cong$ RH irregular isomorphism

$$\mathcal{M}_{B}(\Sigma, \mathcal{C})$$

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$ — Algebraic integrable systems (Hitchin, Nitsure, Bottacin, Markman...)

$\Big|\Big|\Big|$ wild non-abelian Hodge isom.

$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$

$\Big|\Big|\Big|$ RH irregular isomorphism

$\mathcal{M}_{B}(\Sigma, \mathcal{C})$

$$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$$
$$\downarrow$$
$$\mathbb{H}$$

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

$$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$$

$\cong$ wild non-abelian Hodge isom.

$$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$$ — Isomonodromy systems (as Σ varies in admissible fashion)

$\cong$ irregular RH isomorphism

$$\mathcal{M}_{B}(\Sigma, \mathcal{C})$$

$$\begin{array}{c} \underset{\sim}{\Sigma} \\ \downarrow \\ \mathbb{B} \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \mathcal{M}_{DR}(\Sigma_b) & \subset & \mathcal{M}_{DR/\mathbb{B}} \\ \downarrow & & \downarrow \\ b & \in & \mathbb{B} \end{array}$$

— fibre bundle with flat nonlinear connection

e.g. Painlevé equations, Schlesinger system

JMU system, Simply-laced isomonodromy systems

§2 Wild Hitchin moduli spaces

Fix $G = GL_n(\mathbb{C})$

$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$

$|||$ wild non-abelian Hodge isom.

$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$

$|||$ irregular RH isomorphism

$\mathcal{M}_B(\Sigma, \mathcal{C})$ — Nonlinear braid/mapping class group actions

$$\begin{array}{c} \Sigma \\ \downarrow \\ \mathbb{B} \end{array} \Rightarrow \pi_1(\mathbb{B}, b) \circlearrowright \mathcal{M}_B(\Sigma_b)$$

by algebraic Poisson automorphisms

"wild mapping class groups" $= \pi_1(\mathcal{M}(\Sigma))$

e.g. Braiding of Stokes data of Cecotti-Vafa/Dubrovin

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607)

E_8	E_7	E_6	D_4	$A_3 = D_3$	D_2	D_1	D_0	A_2	A_1	A_0
6	4	3	2	2	2	2	2	2	2	2
1+1+1	1+1+1	1+1+1	1+1+1+1	2+1+1	2+2	2+2	2+2	3+1	4	4

affine Weyl group

minimal rank of bundles

pole orders

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(Nonabelian Hodge surfaces) (1203.6607)

Tame ← → Wild

E_8	6	1+1+1
E_7	4	1+1+1
E_6	3	1+1+1
D_4	2	1+1+1+1
$A_3 = D_3$	2	2+1+1
D_2	2	2+2
D_1	2	2+2
D_0	2	2+2
A_2	2	3+1
A_1	2	4
A_0	2	4

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P_3 P_3' P_3''

D_2 D_1 D_0

D_4 $A_3 = D_3$

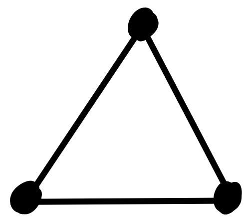
P_6 P_5

A_2 A_1 A_0

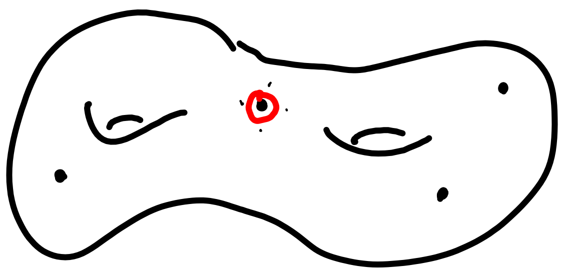
P_4 P_2 P_1

Phase spaces for Painlevé differential equations

§3 Compare / Relate these stories



$$\Rightarrow \quad \mathcal{N}(\Gamma, \underset{\sim}{d}, \underset{\sim}{\lambda})$$



$$\Rightarrow \quad \mathcal{M}(\Sigma, e)$$

§3 Compare / Relate these stories

Suppose $\Sigma = (\mathbb{P}^1, \underset{\sim}{a}, \underset{\sim}{Q})$ rational irreg. curve

Then have open subset $\mathcal{M}^*(\Sigma) \subset \mathcal{M}_{DR}(\Sigma)$

where $V \to \mathbb{P}^1$ holomorphically trivial

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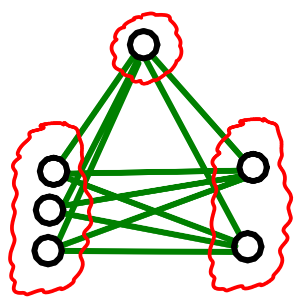
Complete k partite graphs $\Longleftrightarrow$ Integer partitions with k parts



$\Longleftrightarrow$ $1+2+3=6$

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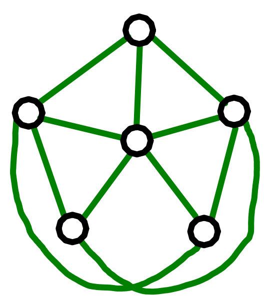

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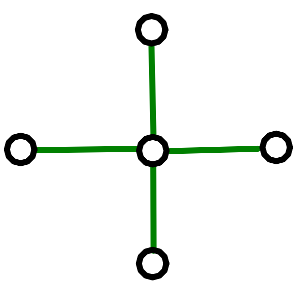
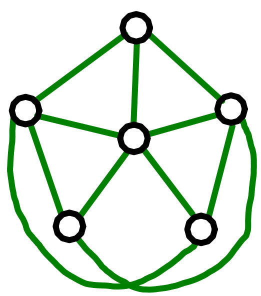


$\Gamma(1,4)$



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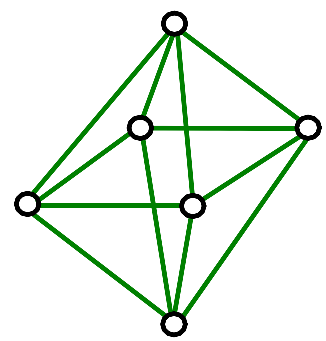
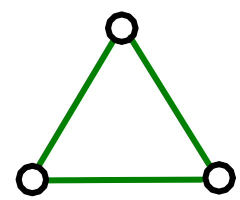


$\Gamma(2,2,2)$



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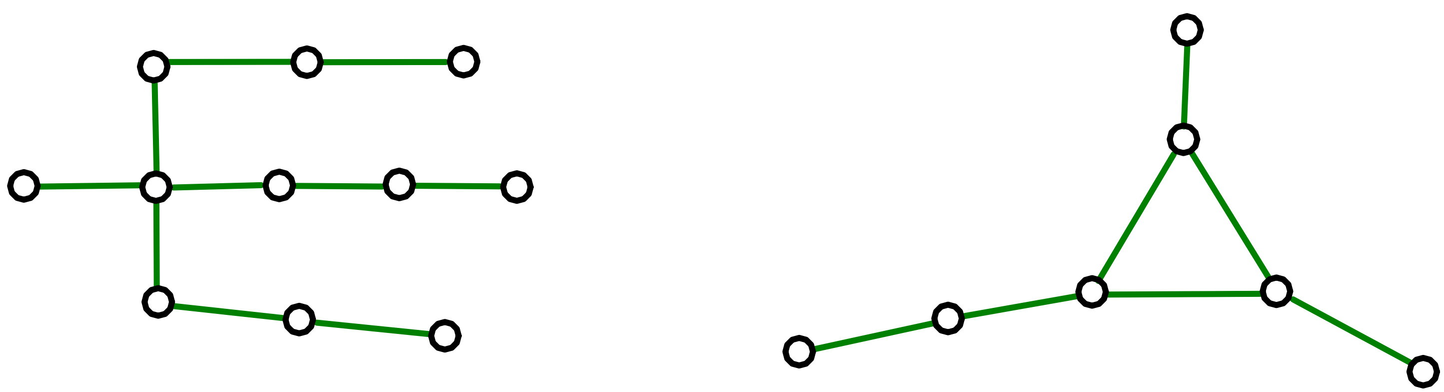
Complete k partite graphs + legs = simply-laced supernova graphs



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[Star-shaped (tame) case due to Nakajima / Crawley-Boevey]

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[Star-shaped (tame) case due to Nakajima / Crawley-Boevey]

- Fourier-Laplace $\Rightarrow$ reflection isom.s
  (integrable system + isomonodromy connection preserved)

"Modular quiver varieties"

Idea/example

$$\Sigma = (\mathbb{P}^1, \infty, Q = Az^2)$$

"Modular quiver varieties"

Idea/example $\quad \Sigma = (\mathbb{P}^1, \infty, Q = Az^2)$

$V = \mathbb{C}^n$, $\quad A \in \mathrm{End}(V)$ diagonal, $\quad V = \bigoplus V_i \quad$ (eigenspaces)

$A = \sum a_i \, \mathrm{Id}_{V_i} \qquad a_i \in \mathbb{C} \qquad$ (eigenvalues)

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for some ! $\Lambda \in H = \bigoplus \mathrm{End}(V_i)$

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complete graph nodes $\{a_i\}$

so $\delta(B) = 0$ & $B \in \bigoplus_{i \neq j} \mathrm{Hom}(V_i, V_j) = \mathrm{Rep}(\Gamma, V)$

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and Λ = moment map for $\mathbb{G} = \prod GL(V_i) \circlearrowright \mathrm{Rep}(\Gamma, V)$

"Modular quiver varieties"

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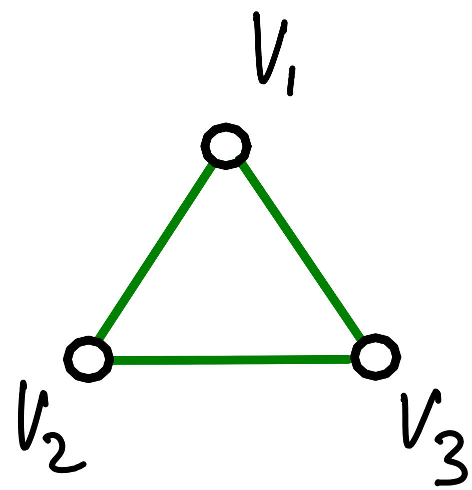


$\dim V_i = d_i$

$\mathrm{Rank} = \sum d_i$

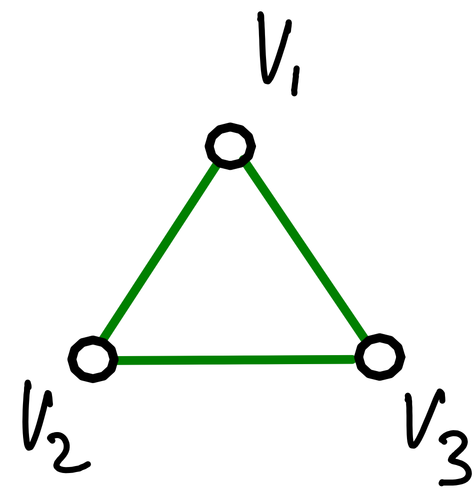
$$B = \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix} \in \mathrm{End}(\oplus V_i)$$

"Modular quiver varieties"

Idea/example

$$\Sigma = (\mathbb{P}^1, \infty, Q = Az^2)$$

$$\Lambda = \begin{pmatrix} \Lambda_1 & & \\ & \Lambda_2 & \\ & & \Lambda_3 \end{pmatrix} \qquad \Lambda_i \in \mathrm{End}(V_i)$$



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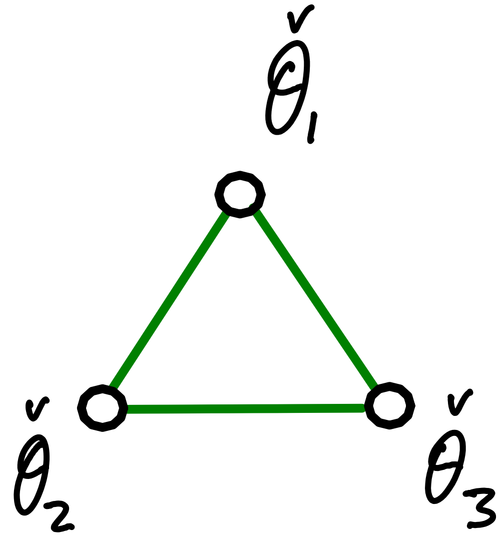

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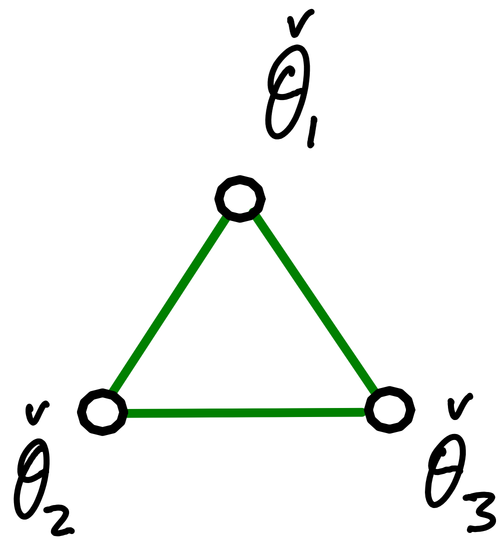

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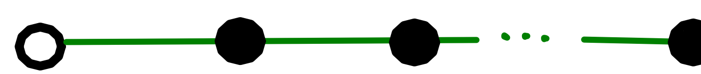
Legs $\iff$ orbits

$\mathcal{O} \subset \mathrm{End}(V) \Rightarrow \mathcal{O} = \mathcal{N}($  $)$

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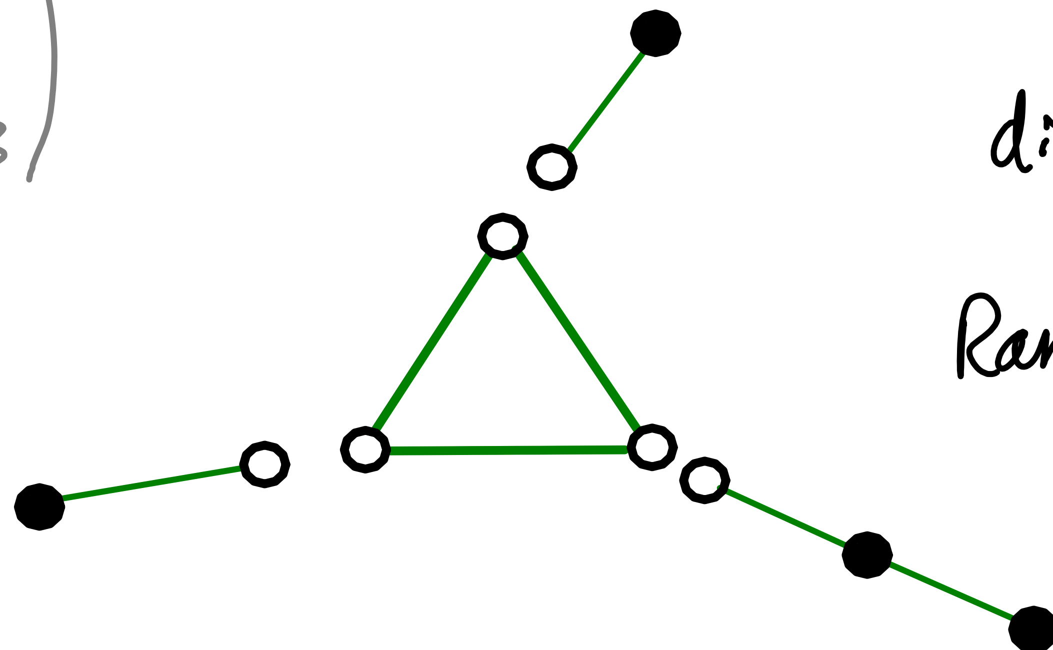

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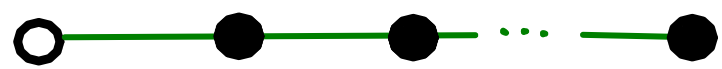
$\mathcal{O} \subset \mathrm{End}(V) \Rightarrow \mathcal{O} = \mathcal{N}($  $)$

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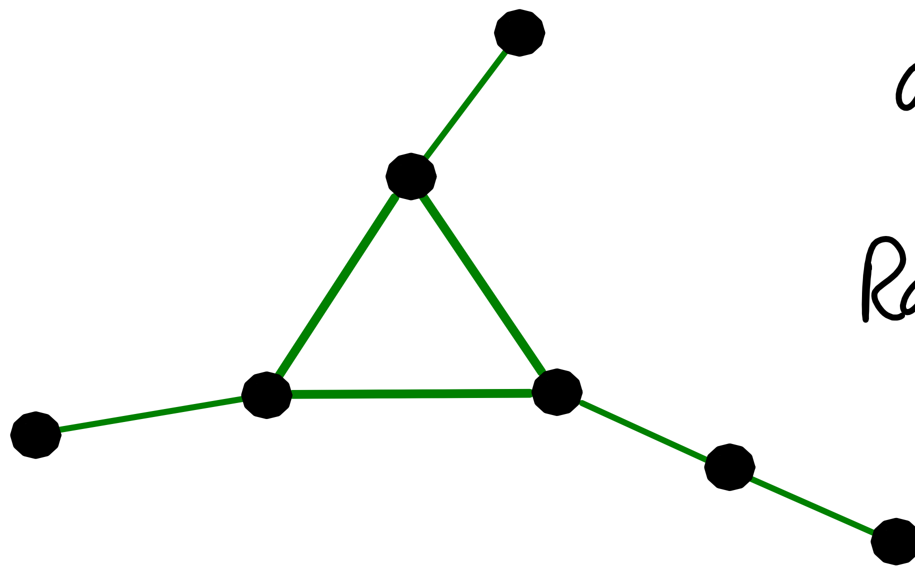

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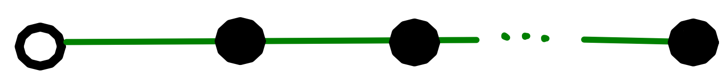
$\mathcal{O} \subset \mathrm{End}(V) \Rightarrow \mathcal{O} = \mathcal{N}($  $)$

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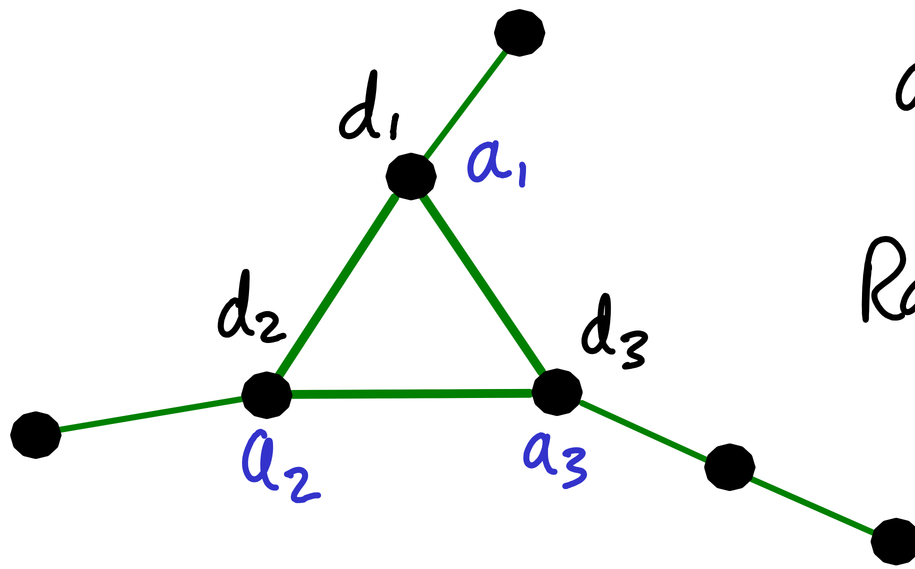

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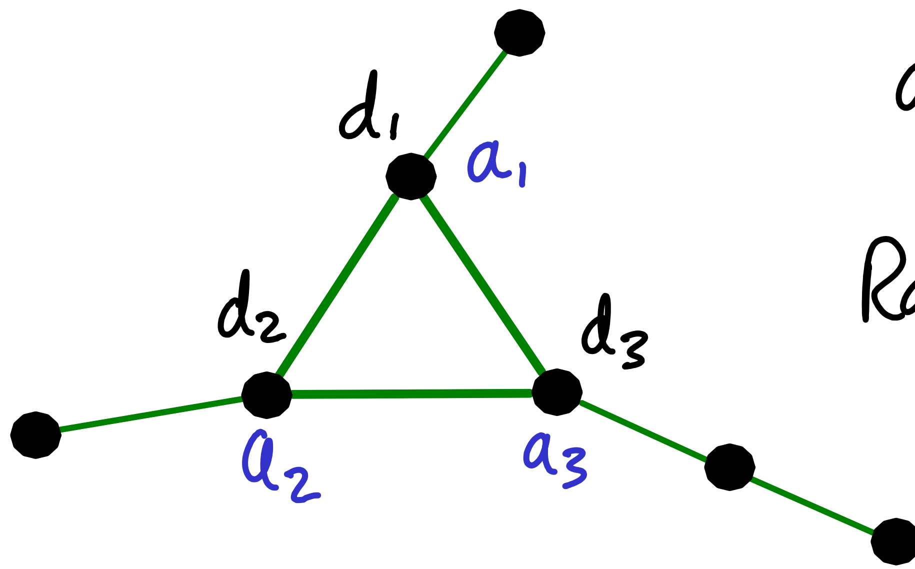

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Fourier-Laplace changes eigenvalues of A

$$a_i \longmapsto -1/a_i$$

"Modular quiver varieties"

Idea/example

Suppose $a_1 = \infty$

$$\Sigma = \left(\mathbb{P}^1, (0,\infty), (0, Az^2)\right) \quad \begin{cases} \text{Rank} = d_2 + d_3 \\ A = a_2 \mathrm{Id}_{V_2} + a_3 \mathrm{Id}_{V_3} \end{cases}$$

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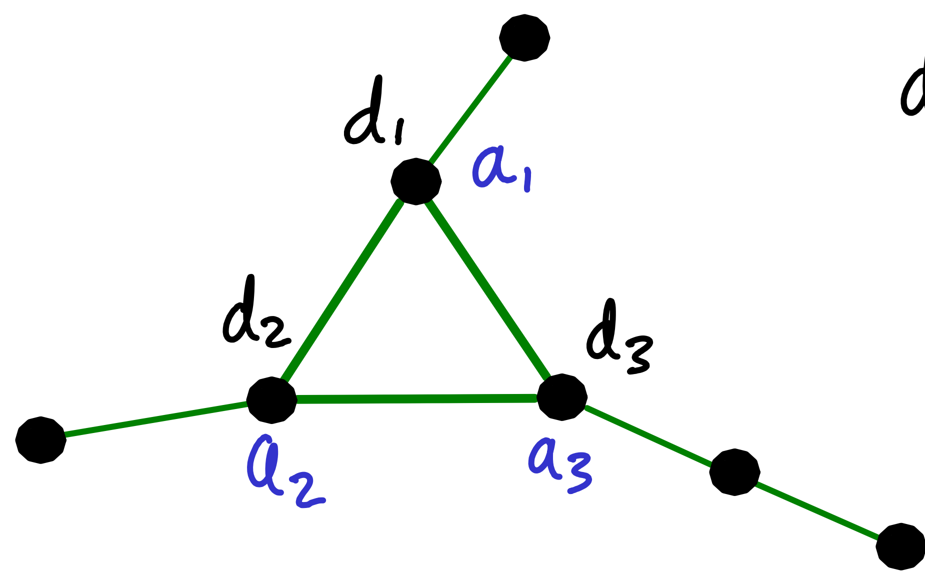

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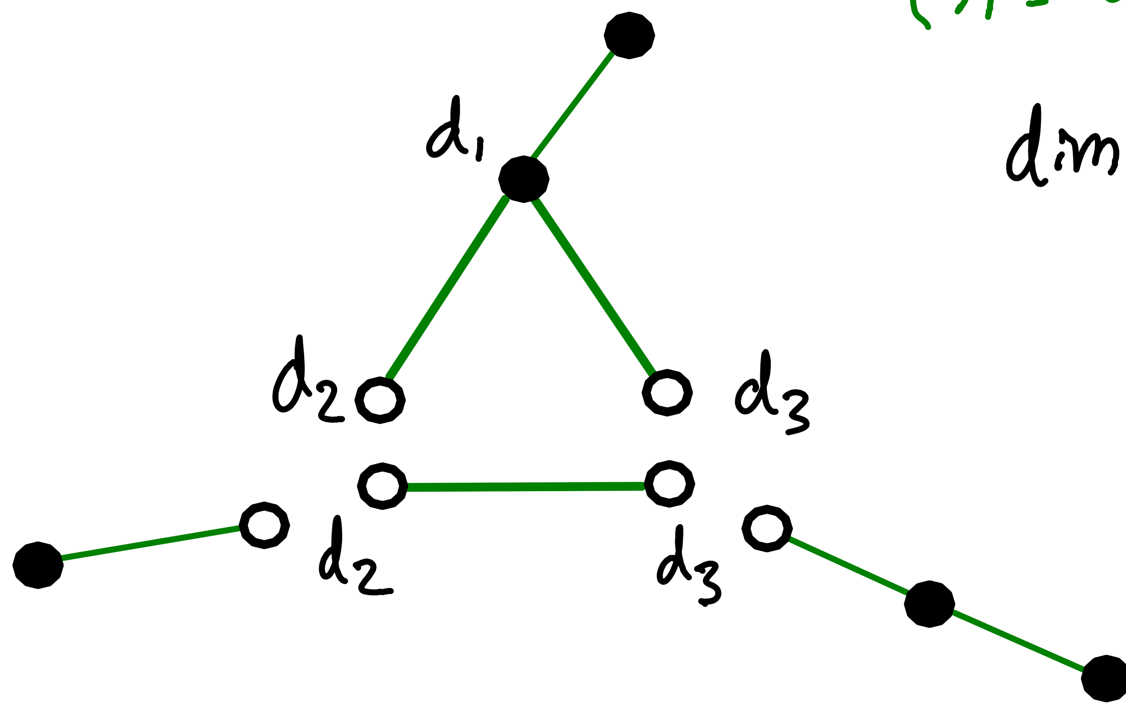

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orbit of residue at 0

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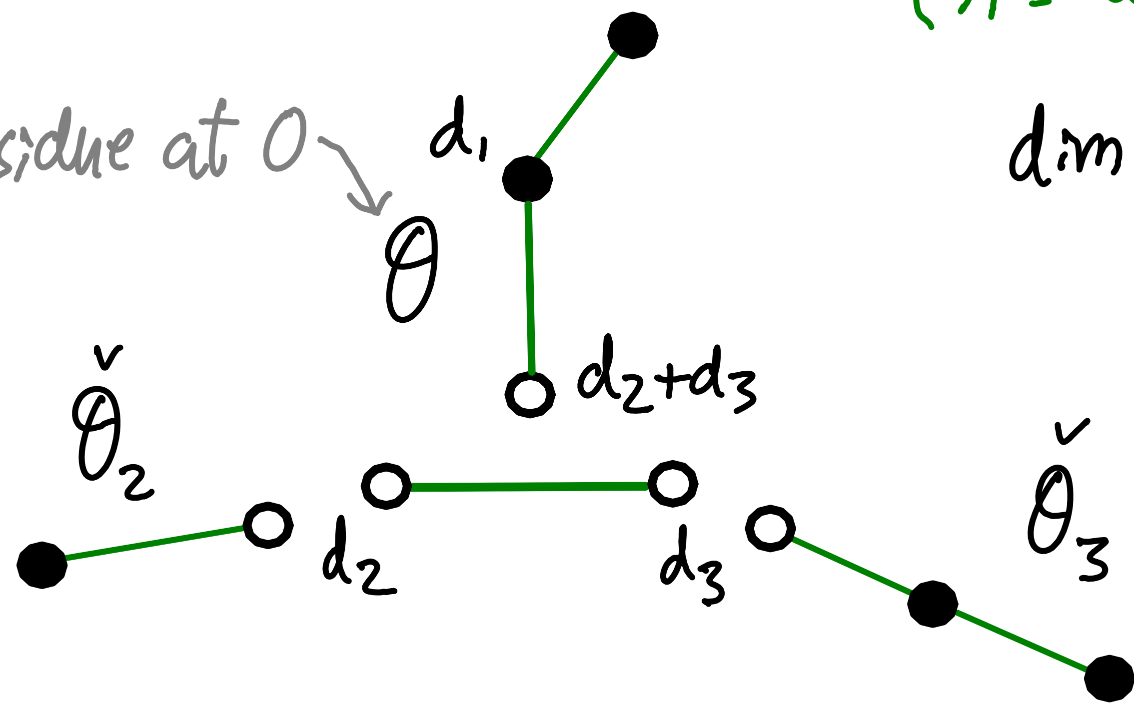

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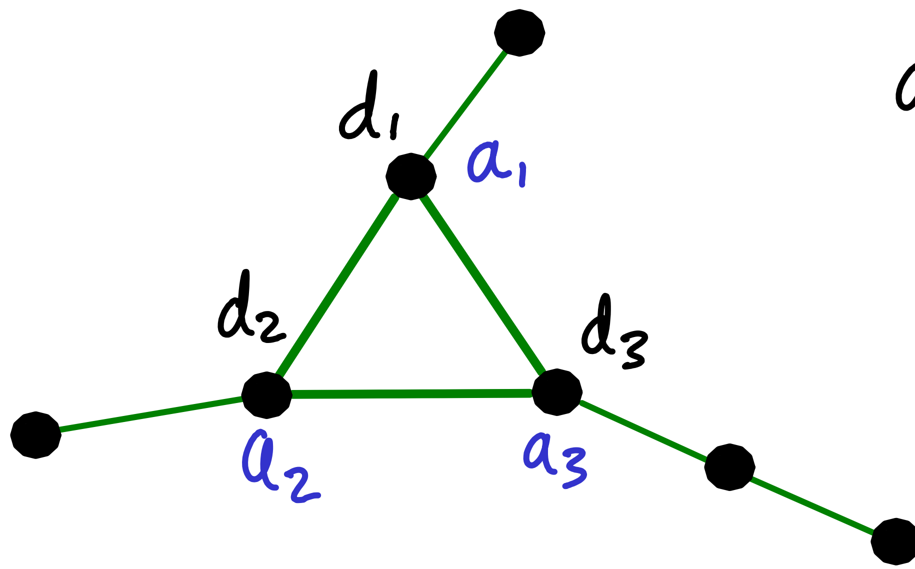

$\rightsquigarrow$ Dictionary "k+1 ways to read a complete k-partite graph as moduli of connections"

Question:

Are the corresponding full moduli spaces isomorphic?

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$$\begin{tikzcd}
 & \mathcal{M}^*(\Sigma_1, e_1) \\
\mathcal{N}(\Gamma, \underset{\sim}{\lambda}, \underset{\sim}{d}) \arrow[ur, dash, "\cong"] \arrow[dr, dash, "\cong"'] & \\
 & \mathcal{M}^*(\Sigma_2, e_2)
\end{tikzcd}$$

Question:

Are the corresponding full moduli spaces isomorphic?

$$
\begin{tikzcd}
 & \mathcal{M}^*(\Sigma_1, e_1) \arrow[d, phantom, "\equiv"] \\
\mathcal{N}(\Gamma, \underset{\sim}{\lambda}, \underset{\sim}{d}) \arrow[ur, equal] \arrow[dr, equal] & \\
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\end{tikzcd}
$$

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$$\begin{tikzcd}
 & \mathcal{M}^*(\Sigma_1, e_1) \subset \mathcal{M}_{DR}(\Sigma_1, e_1) \arrow[dd, phantom, "\equiv"] \\
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\begin{tikzcd}
 & \mathcal{M}^*(\Sigma_1, e_1) \arrow[dd, equal] & \subset & \mathcal{M}_{DR}(\Sigma_1, e_1) \arrow[dd, equal, "\exists"', "?"] \\
\mathcal{N}(\Gamma, \underset{\sim}{\lambda}, \underset{\sim}{d}) \arrow[ur, equal] \arrow[dr, equal] & & & \\
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\mathcal{N}(\Gamma, \underset{\sim}{\lambda}, \underset{\sim}{d}) \arrow[ur, phantom, sloped, "\cong"] \arrow[dr, phantom, sloped, "\cong"] & \\
 & \mathcal{M}^*(\Sigma_2, e_2) \subset \mathcal{M}_{DR}(\Sigma_2, e_2)
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$$

Thm ('13)

There is a new theory of "multiplicative quiver varieties" and these may be read (similarly) as wild character varieties

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$$
\begin{tikzcd}[column sep=small]
 & \mathcal{M}^*(\Sigma_1,e_1) \arrow[dd, phantom, "\cong" description] & \subset & \mathcal{M}_B(\Sigma_1,e_1) \arrow[dd, phantom, "\cong\ \text{alg. sympl.}" description] & \\
\mathcal{N}(\Gamma,\underset{\sim}{\lambda},\underset{\sim}{d}) \arrow[ur, phantom, "\cong"] \arrow[dr, phantom, "\cong"] & & & & \mathcal{M}(\Gamma,\underset{\sim}{q},\underset{\sim}{d}) \arrow[ul, phantom, "\cong"] \arrow[dl, phantom, "\cong"] \\
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\end{tikzcd}
$$

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There is a new theory of "multiplicative quiver varieties" and these may be read (similarly) as wild character varieties

Wild character varieties

$$\mathcal{M}_B(\Sigma, \mathcal{C}) \cong \{\text{Stokes local systems with formal monodromy } \mathcal{C}\}/\text{isom.}$$

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Idea/example

$$\Sigma = (\mathbb{P}^1, \infty, Q = Az^2)$$

$V = \mathbb{C}^n$, $\quad A \in \mathrm{End}(V)$ diagonal, $\quad V = \oplus V_i$ (eigenspaces)

$A = \sum a_i \, \mathrm{Id}_{V_i}$ $\quad a_i \in \mathbb{R}$ (eigenvalues)

ordered

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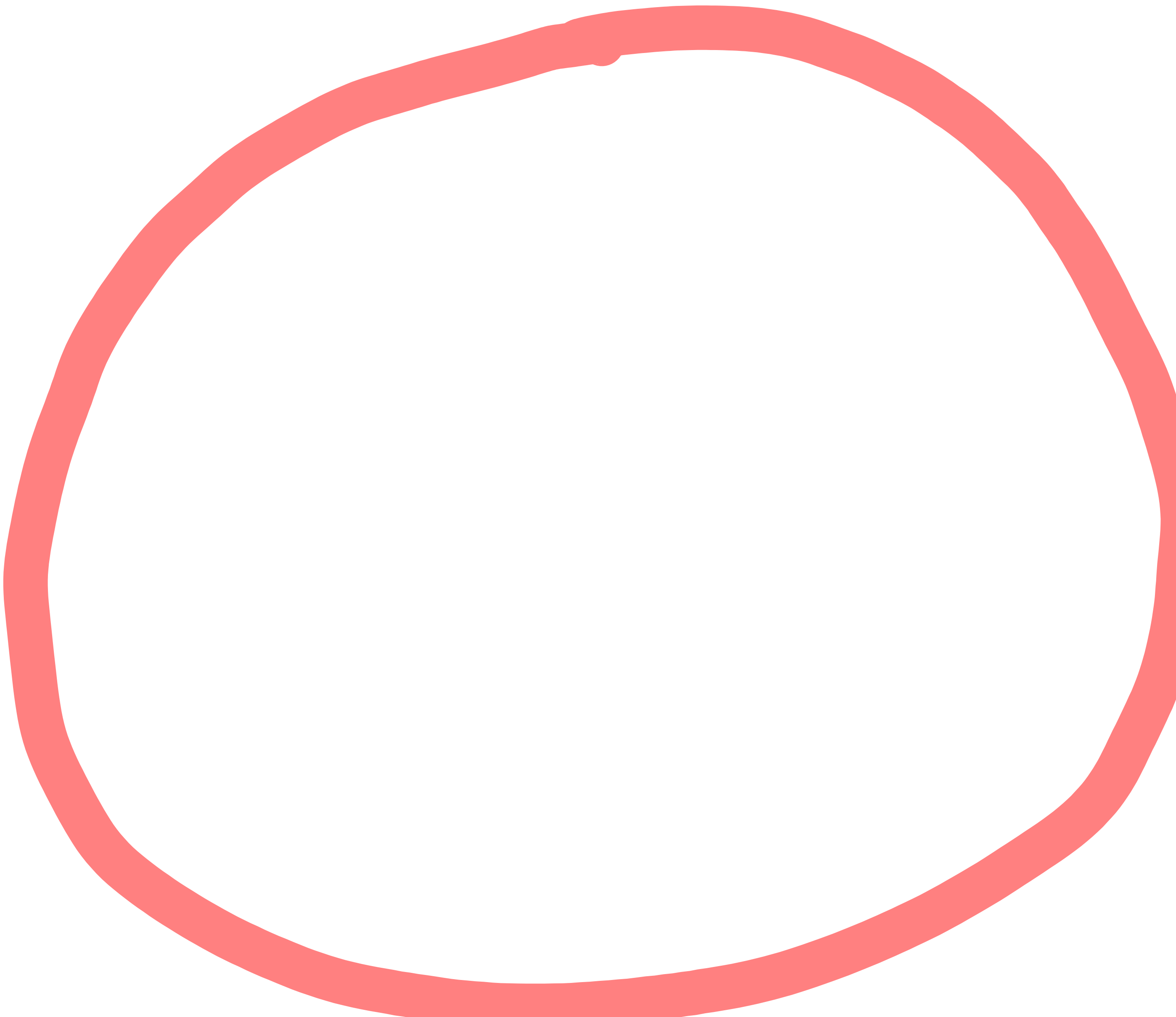
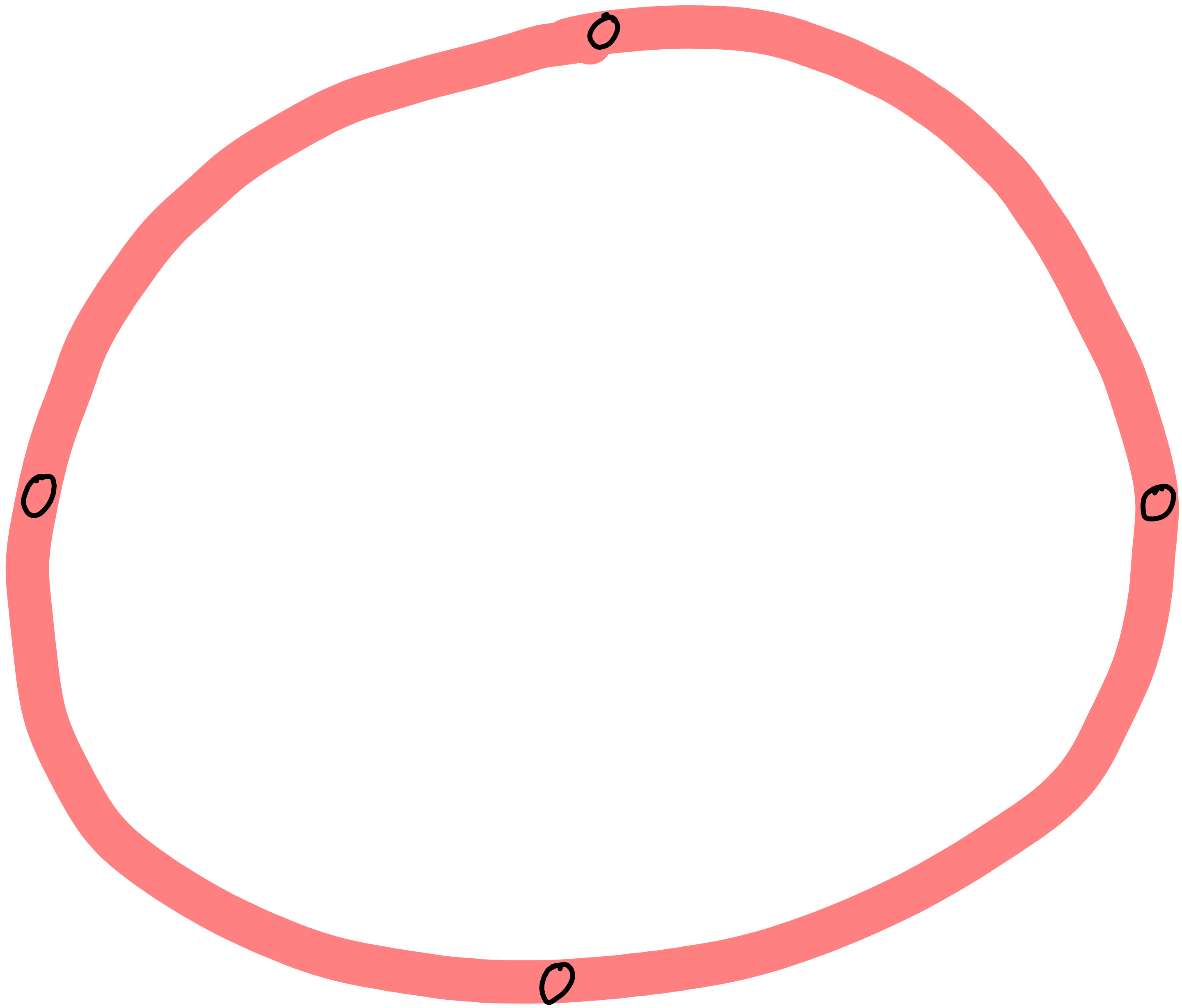
ordered

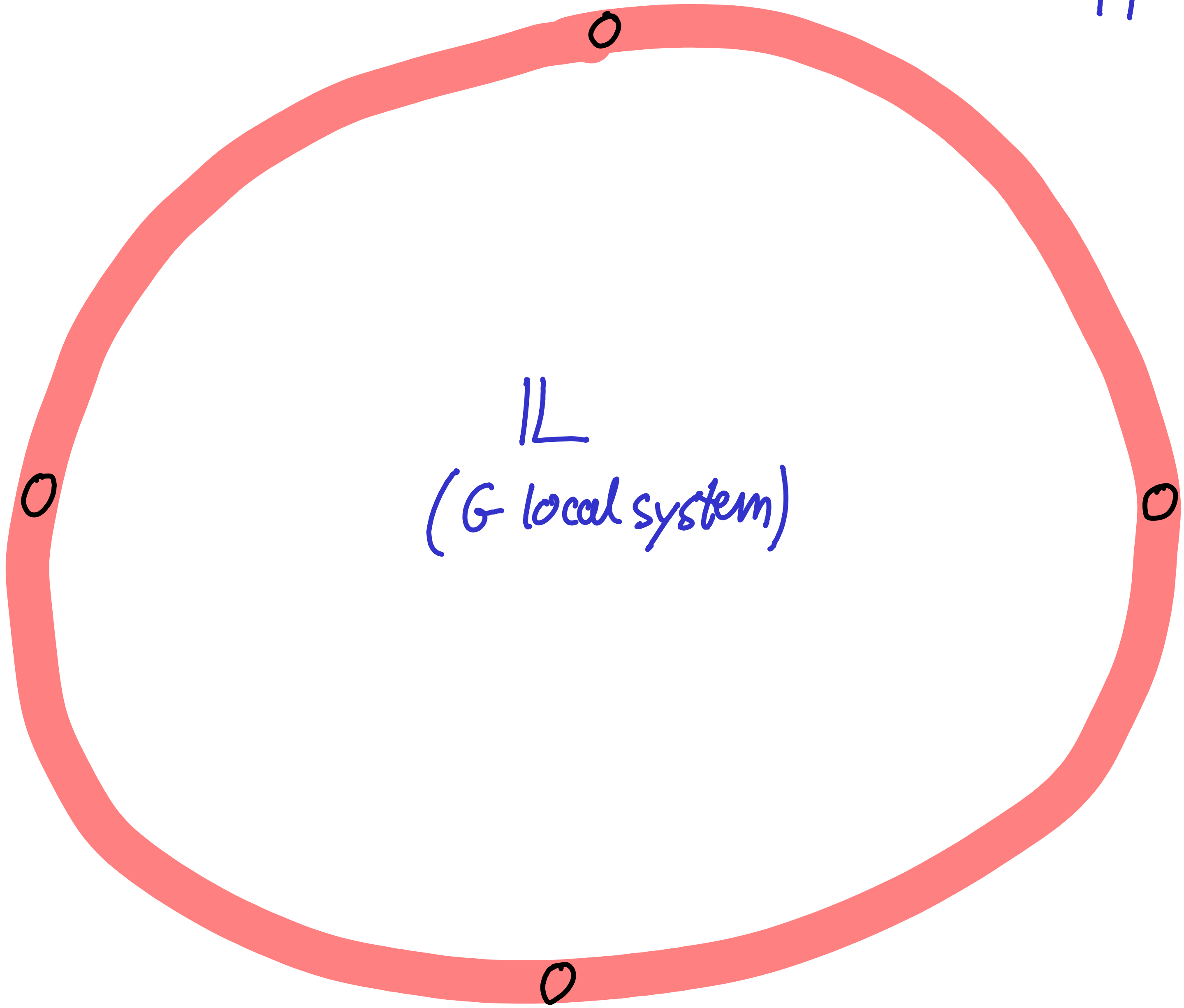
$$H = \prod GL(V_i) \subset G = GL(V) \supset U_+ = \left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right\}$$

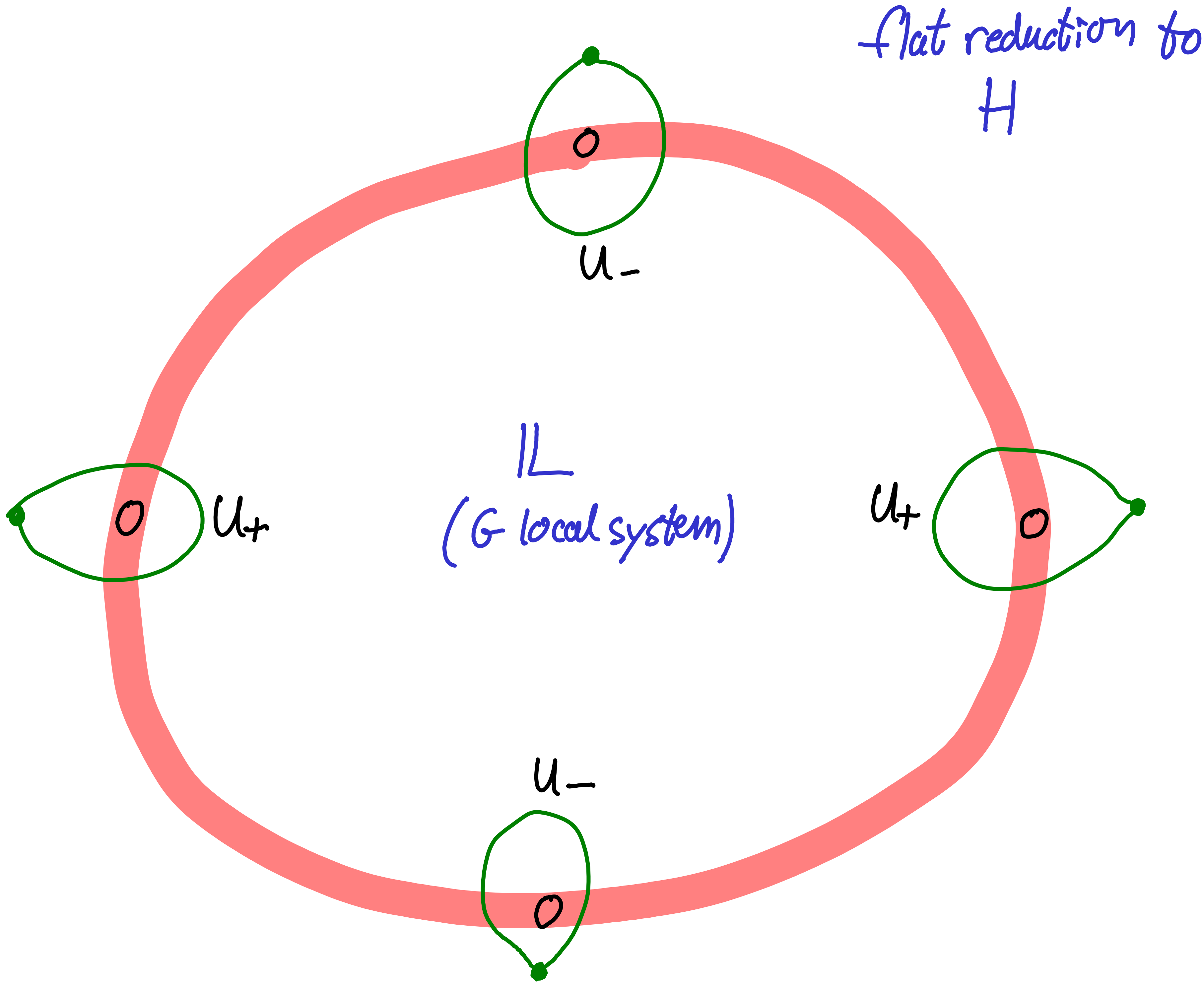
$$U_- = \; = \left\{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}\right\}$$

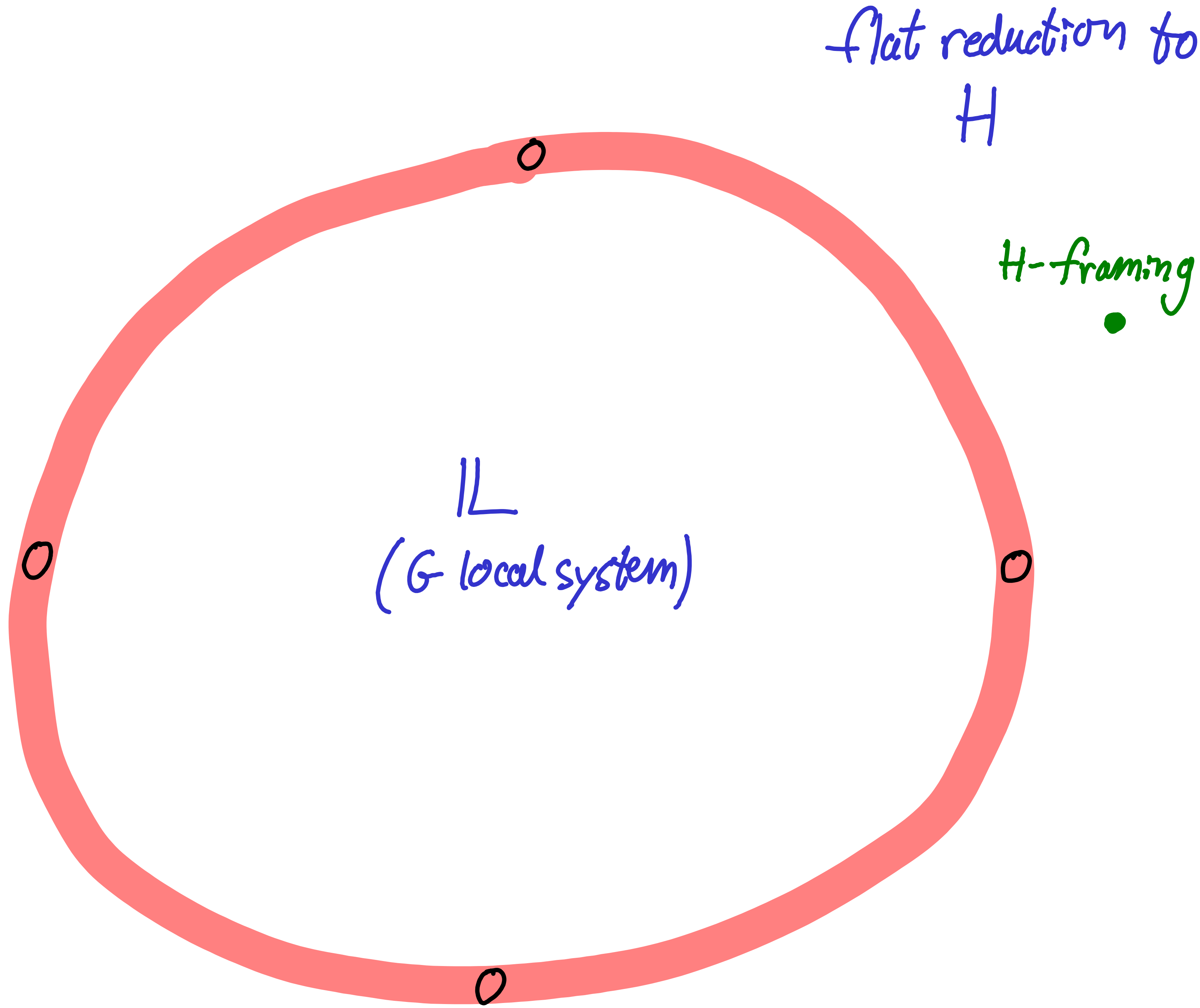




flat reduction to H




flat reduction to H
$\mathcal{O}$
U_-
$\mathcal{O}$
U_+
$\mathbb{L}$
(G local system)
U_+
$\mathcal{O}$
U_-
$\mathcal{O}$


flat reduction to
H
H-framing
𝕃
(G local system)

$$\{\text{H-framed Stokes local systems}\}/\text{isom.}$$

$$\cong \left\{ (S_1, S_2, S_3, S_4, h) \in U_+ \times U_- \times U_+ \times U_- \times H \;\middle|\; h S_4 S_3 S_2 S_1 = 1 \right\}$$

$$\{\text{H-framed Stokes local systems}\}/\text{isom.}$$

$$\cong \left\{ (S_1, S_2, S_3, S_4, h) \in U_+ \times U_- \times U_+ \times U_- \times H \;\middle|\; h S_4 S_3 S_2 S_1 = 1 \right\}$$

Thm ('02, '11) This is a quasi-Hamiltonian H-space
(moment map = h^{-1})

$\{$ H-framed Stokes local systems $\}/\text{isom.}$

$$\cong \{(S_1, S_2, S_3, S_4, h) \in U_+ \times U_- \times U_+ \times U_- \times H \mid h S_4 S_3 S_2 S_1 = 1\}$$

Thm ('02, '11) This is a quasi-Hamiltonian H-space (moment map = h^{-1})

Lemma It is open in $\mathrm{Rep}(\Gamma, V)$ (Γ complete graph)

$$\mathrm{Rep}(\Gamma, V) \cong \{(S_2, S_1)\} = U_- \times U_+$$

$\{$ H-framed Stokes local systems $\}/$isom.

$$\cong \left\{ (S_1, S_2, S_3, S_4, h) \in U_+ \times U_- \times U_+ \times U_- \times H \,\middle|\, h S_4 S_3 S_2 S_1 = 1 \right\}$$

Thm ('02, '11) This is a quasi-Hamiltonian H-space (moment map $= h^{-1}$)

Lemma It is open in $\mathrm{Rep}(\Gamma, V)$ (Γ complete graph)

Call it

$$\begin{array}{ccc} \mathrm{Rep}^*(\Gamma, V) & \subset & \mathrm{Rep}(\Gamma, V) \\ \downarrow \mu & & \downarrow \mu \\ H & & \mathfrak{h}^* \end{array}$$

"invertible rep.s"

$$\begin{array}{ccc} \mathrm{Rep}^*(\Gamma, V) & \subset & \mathrm{Rep}(\Gamma, V) \\ \downarrow \mu & & \downarrow \mu \\ H & & \mathfrak{h}^* \end{array}$$

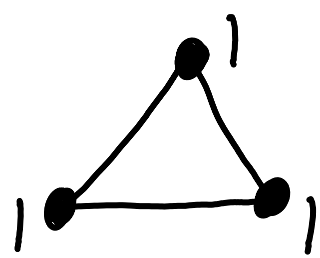
<u>Cor.</u> $\mathcal{M}_B(\Sigma, e) \cong \mathrm{Rep}^*(\Gamma, V) /\!/_{e} H$

(then e ~ gluing on legs etc as before)

$$\begin{array}{ccc} \mathrm{Rep}^*(\Gamma, V) & \subset & \mathrm{Rep}(\Gamma, V) \\ \downarrow \mu & & \downarrow \mu \\ H & & H^* \end{array}$$

<u>Cor.</u> $\mathcal{M}_B(\Sigma, \mathcal{C}) \cong \mathrm{Rep}^*(\Gamma, V) /\!\!/_{\mathcal{C}} H$

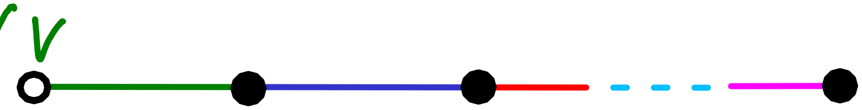
(then $\mathcal{C}$ ~ gluing on legs etc as before)

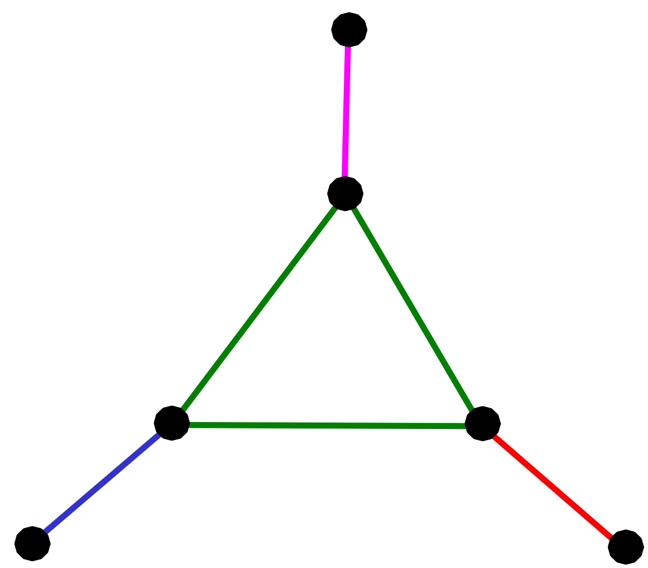
e.g multiplicative quiver var. of 

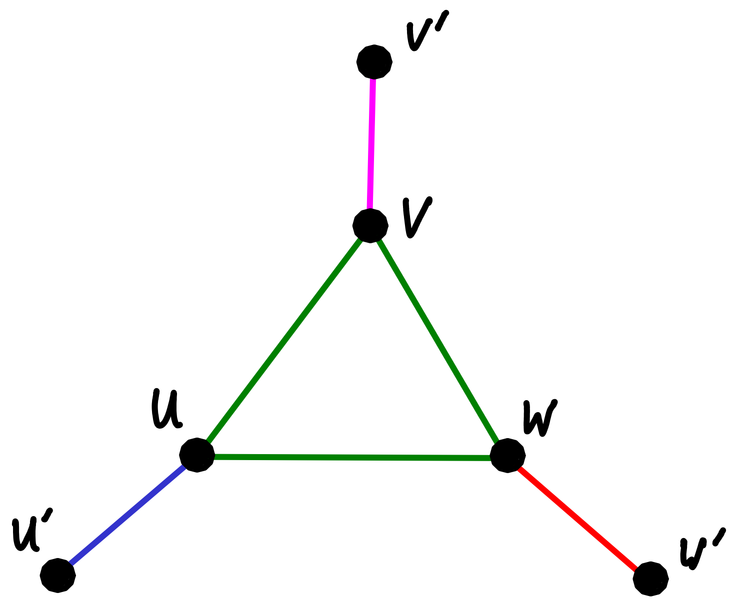
is $\cong$ new hyperkähler 4 manifold $\not\cong$ ALE space

Recall (Kraft–Procesi / Nakajima / Crawley-Boevey + Shaw)

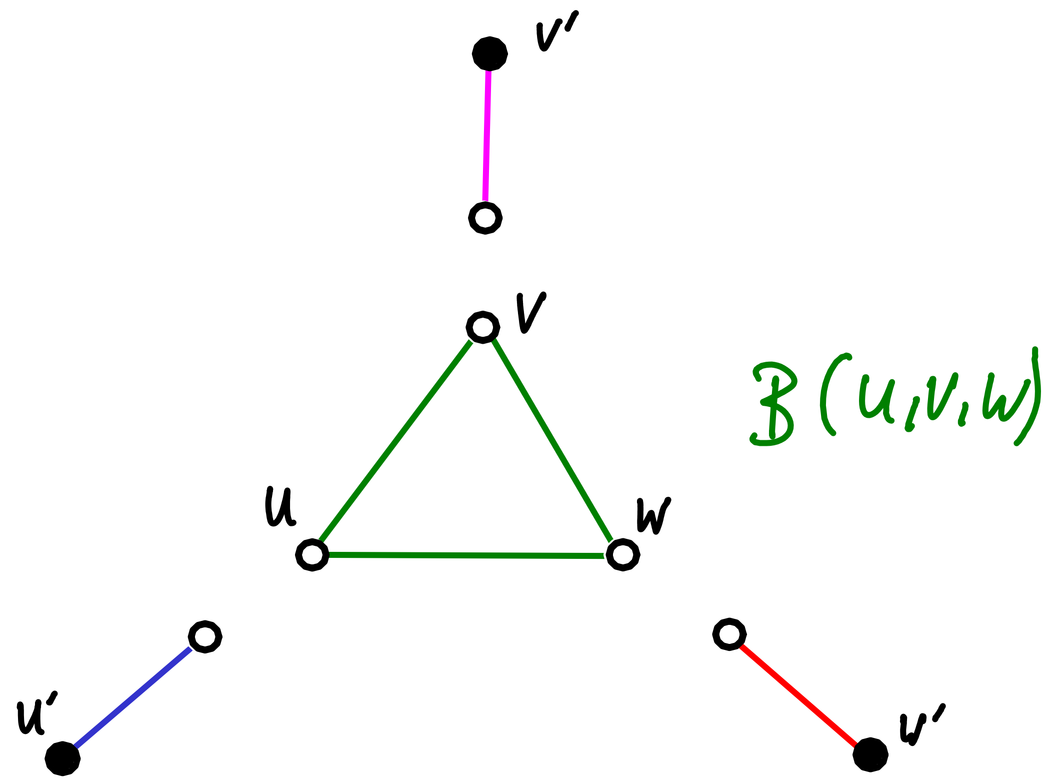
conjugacy classes $\mathcal{C} \subset GL(V)$ are classical mult. quiver varieties:

$\mathcal{C} \cong \mathrm{MQV}\Big($  $\Big)$



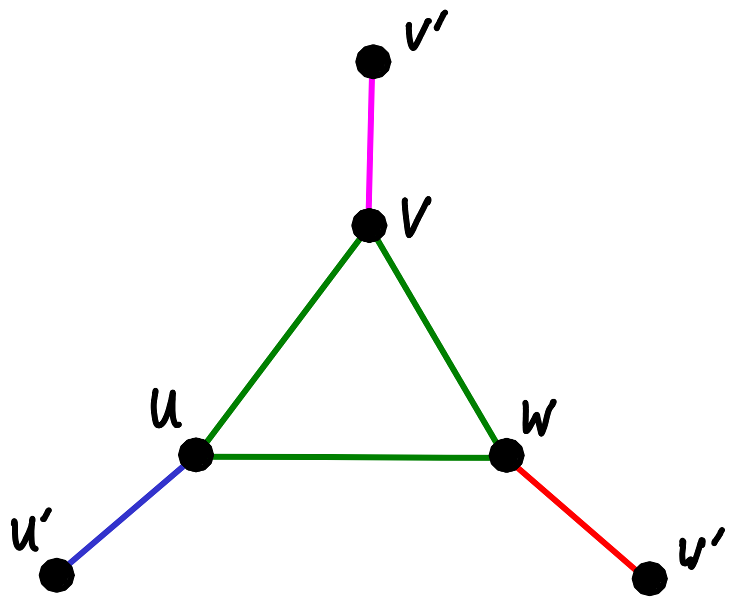


v'
v
u
w
u'
w'



Pole order(s) Rank

3 $\dim(U \oplus V \oplus W)$



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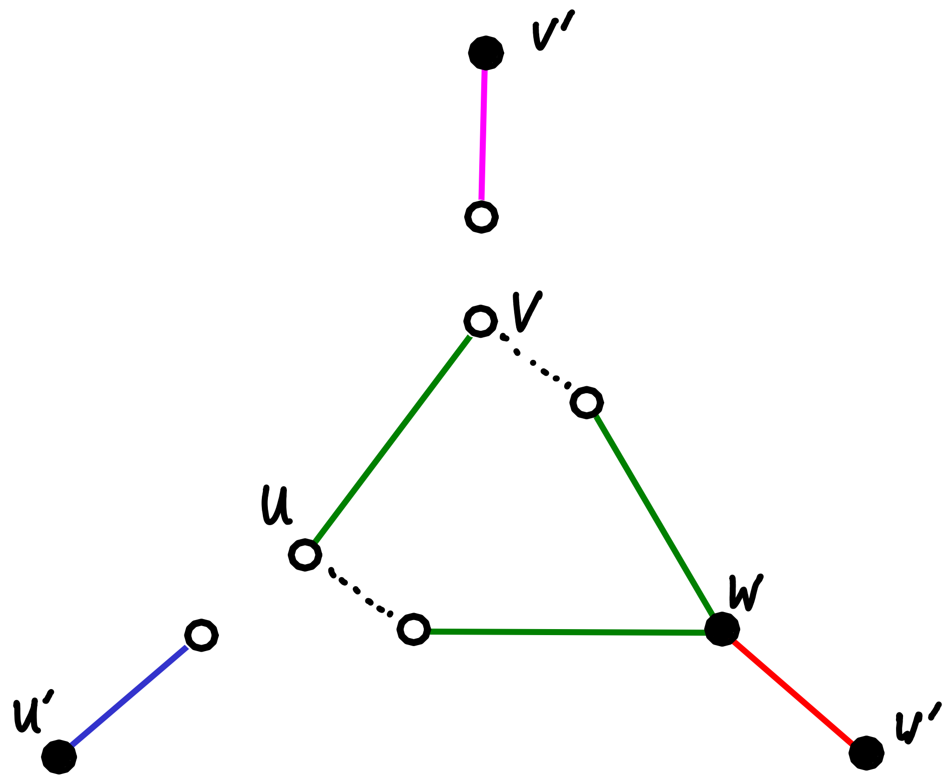
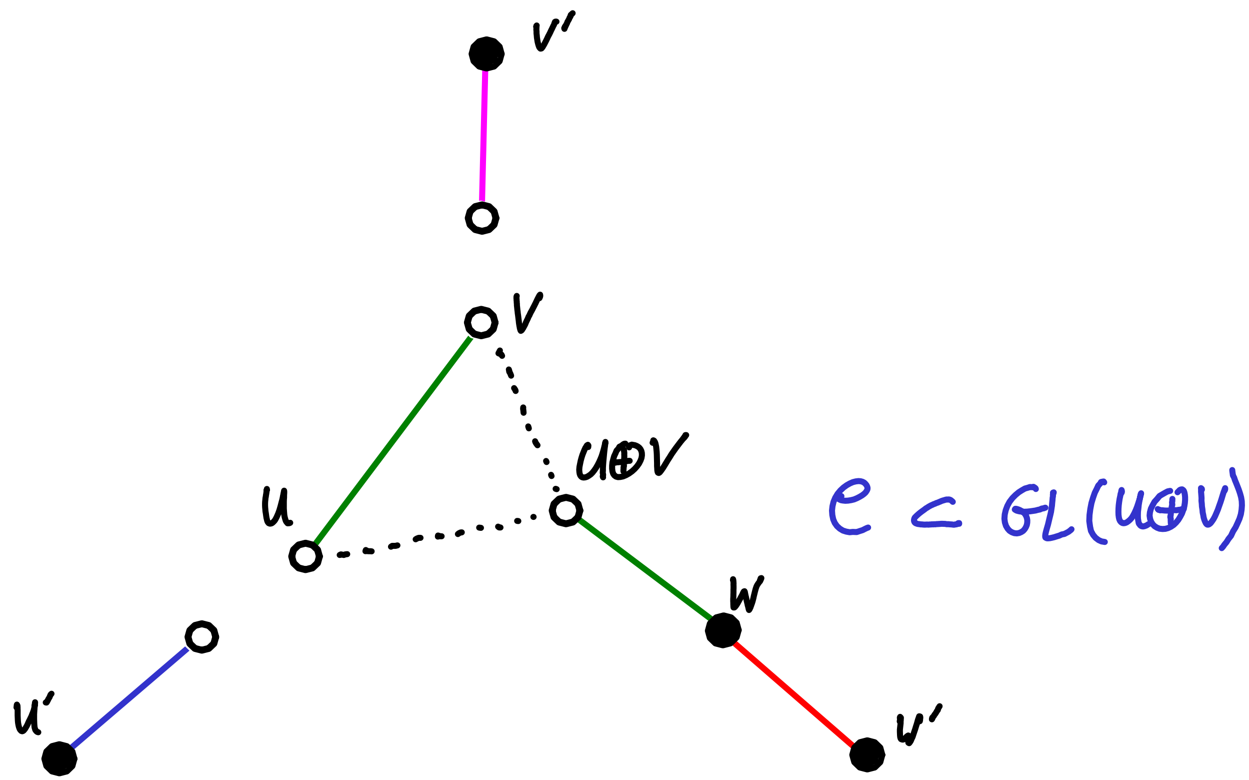


Pole order(s)	Rank
$3 + 1$	$\dim(U \oplus V)$



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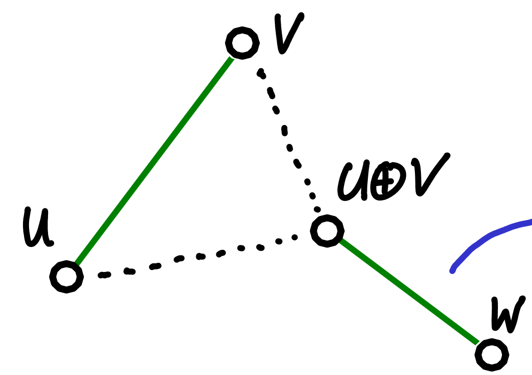


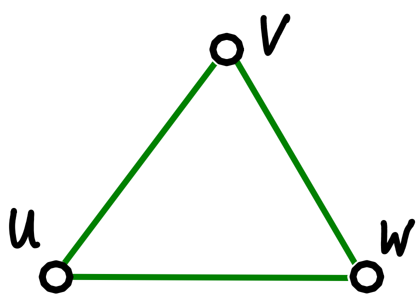
$\mathcal{B}(u\oplus v,\, w)$

$$\mathcal{A}^2(u,v) \underset{G}{\succ\!\!-\!\!\prec} \mathcal{B}(u\oplus v,\, w)$$

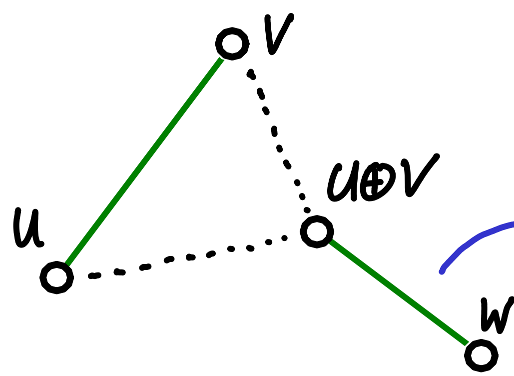
1-fission operator $= {}_H\mathcal{A}^2_G \cong H \times (U_+ \times U_-)^2 \times G$

$$\left\{\begin{array}{l} G = GL(u\oplus v) \\ H = GL(u)\times GL(v) \end{array}\right\}$$

Thm (§7 1307.1033)



$\cong$



$\mathcal{B}(U\oplus V, W)$

$$\mathcal{B}(U,V,W) \cong \mathcal{A}^2(U,V) \underset{G}{)\!-\!(} \mathcal{B}(U\oplus V, W)$$

(Alg. isom. of q-Ham'n spaces)

1-fission operator $= {}_H\mathcal{A}^2_G \cong H \times (U_+ \times U_-)^2 \times G$

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