Non-Perturbative Hyperkähler Manifolds

P. Boalch (CNRS & Orsay)

cf. also surveys arXiv: 1203 (Hyperkähler)
1309 (Poisson)

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meromorphic Hitchin systems

Boalch, Markman 1995

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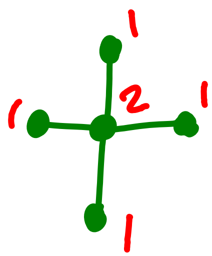

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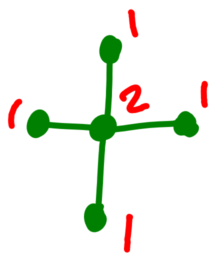
Eg 

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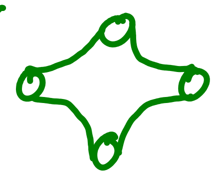
$\mathcal{M}($  $, GL_2)$

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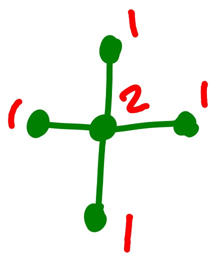

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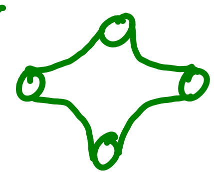
$\mathcal{M}($  $, GL_2)$

$\cong$

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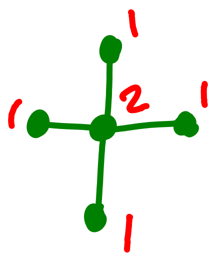

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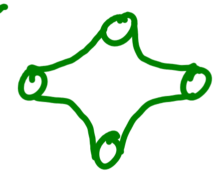
$\mathcal{M}\big($  $, GL_2\big)$

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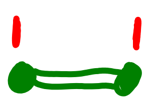
What about 

$\Longrightarrow$ Eguchi-Hanson $\subset$?

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$$\left(\frac{A}{z^2} + \frac{B}{z}\right)dz \Longrightarrow \text{Poisson Lie group underlying } U_q \mathfrak{g}$$

$$\mathbb{C}[\mathfrak{g}^*] = \mathrm{Sym}\,\mathfrak{g} \rightsquigarrow U\mathfrak{g}$$

quantize

(deform multiplication)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
 & U_q\mathfrak{g} \\
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Thm (2001) G^* is the space of monodromy/Stokes data of connections $\left(\frac{A}{z^2} + \frac{B}{z}\right) dz \Big|_{\text{unit disc}}$ $\quad A \in \mathfrak{t}_{reg}$ fixed, $B \in \mathfrak{g} \cong \mathfrak{g}^*$

and the desired nonlinear Poisson structure appears this way

Cartoon

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Hamiltonian geometry

$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

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$\mathcal{O} \subset \mathfrak{g}^*, \quad T^*G$

$$\Big\downarrow \ \mu^{-1}(0)/G$$

Additive symplectic geometry

$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

Cartoon

∞-d Ham^n geometry

e.g connections on C^∞ bundles / Riemann surfaces

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Multiplicative symplectic geometry

Betti spaces, character varieties

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mult. sp. quotient $\mu^{-1}(1)/G$

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$$
\begin{tikzcd}
 & \begin{array}{c}\infty\text{-d Ham}^{n}\text{ geometry}\\ \text{e.g. connections on } C^{\infty} \text{ bundles / Riemann surfaces}\end{array} \arrow[dl, phantom, "\cup"] \arrow[dr, squiggly, "/\!/\mathcal{G}_1"] & \\
\begin{array}{c}\text{Hamiltonian geometry}\\ \mathcal{O} \subset \mathfrak{g}^{*}, \quad T^{*}G\end{array} \arrow[d, squiggly, "\mu^{-1}(0)/G"] & & \begin{array}{c}\text{quasi-Hamiltonian geometry}\\ \mathcal{C} \subset G, \quad D = G \times G\end{array} \arrow[d, squiggly, "\mu^{-1}(1)/G", "\text{mult. sp. quotient}"'] \\
\begin{array}{c}\text{Additive symplectic geometry}\\ \mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G\end{array} \arrow[rr, Rightarrow, "RH"] & & \begin{array}{c}\text{Multiplicative symplectic geometry}\\ \text{Betti spaces, character varieties}\end{array}
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$\mathcal{M}_B$

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$\xrightarrow{\text{RHB}}$

Multiplicative symplectic geometry
Betti spaces, wild character varieties

$\mathcal{M}_B$

Van den Bergh's spaces

$$\Gamma = \circ\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ$$

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$\Gamma =$  $\qquad I = \{\text{nodes}(\Gamma)\}$

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$I = \{\text{nodes}(\Gamma)\}$

$V = V_1 \oplus V_2$ (I graded complex vector space)

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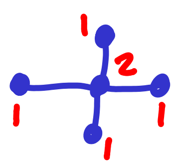
E.g. Mult. Quiver Var. () $\cong \{ xyz + x^2 + y^2 + z^2 = ax + by + cz + d \}$

Qn Suppose $\Gamma =$  or  etc

then what is $\mathrm{Rep}^*(\Gamma, V)$?

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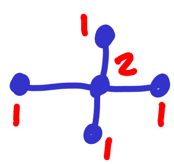
E.g. Mult. Quiver Var. () $\cong \{ xyz + x^2 + y^2 + z^2 = ax + by + cz + d \}$

SPECIMEN ALGORITHMI SINGVLARIS.

Auctore
L. EVLERO.

1.

Consideratio fractionum continuarum, quarum vsum vberrimum per totam Analysin iam aliquoties ostendi, deduxit me ad quantitates certo quodam modo ex indicibus formatas, quarum natura ita est comparata, vt singularem algorithmum requirat. Cum igitur summa Analyseos inuenta maximam partem algorithmo ad certas quasdam quantitates accommodato

6. Haec ergo teneatur definitio ſignorum ( ), inter quae indices ordine a ſiniſtra ad dextram ſcribere conſtitui ; atque indices hoc modo clauſulis incluſi inpoſterum denotabunt numerum ex iſtis indicibus formatum. Ita a ſimpliciſſimis caſibus inchoando, habebimus :

$$(a) = a$$
$$(a,b) = ab + 1$$
$$(a,b,c) = abc + c + a$$
$$(a,b,c,d) = abcd + cd + ad + ab + 1$$
$$(a,b,c,d,e) = abcde + cde + ade + abe + abc + e + c + a$$

etc.

ex

"Euler's continuant polynomials"

G. G. Stokes 1857

VI. *On the Discontinuity of Arbitrary Constants which appear in Divergent Developments.* *By* G. G. STOKES, M.A., D.C.L., Sec. R.S., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

[Read *May* 11, 1857.]

IN a paper "On the Numerical Calculation of a class of Definite Integrals and Infinite Series," printed in the ninth volume of the *Transactions* of this Society, I succeeded in developing the integral $\int_0^\infty \cos\frac{\pi}{2}(w^3 - mw)\,dw$ in a form which admits of extremely easy numerical calculation when m is large, whether positive or negative, or even moderately large. The method there followed is of very general application to a class of functions which frequently occur in physical problems. Some other examples of its use are given in the same paper; and I was enabled by the application of it to solve the problem of the motion of the fluid surrounding a pendulum of the form of a long cylinder, when the internal friction of the fluid is taken into account *.

These functions admit of expansion, according to ascending powers of the variables, in series which are always convergent, and which may be regarded as defining the functions for all values of the variable real or imaginary, though the actual numerical calculation would involve a labour increasing indefinitely with the magnitude of the variable. They satisfy certain linear differential equations, which indeed frequently are what present themselves in the first instance, the series, multiplied by arbitrary constants, being merely their integrals. In my former paper, to which the present may be regarded as a supplement, I have employed these equations to obtain integrals in the form of descending series multiplied by exponentials. These integrals, when once the arbitrary constants are determined, are exceedingly convenient

Wild Character Varieties

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Fix G (e.g $GL_n(\mathbb{C})$)

symplectic variety

Σ compact Riemann Surface $\Rightarrow$ $\mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G$

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$$\Big\| \wr \ RH$$

$\mathcal{M}_{DR} = \{\text{Alg. connections on } G\text{-bundles on } \Sigma\}/\text{isom}$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

symplectic variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \ldots, a_m)$ $\Longrightarrow$ $\mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G$

$\Big\| \wr$ RH

$\mathcal{M}_{DR} = \{\text{Alg. connections on } G\text{-bundles on } \Sigma\}/\text{isom}$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Poisson variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ $\Longrightarrow$ $\mathcal{M}_B^{tame} = \mathrm{Hom}(\pi_1(\Sigma^0), G)/G$

$\Sigma^0 = \Sigma \setminus \underset{\sim}{a}$

$\cong$ RH

$\mathcal{M}_{DR} = \{$ Alg. connections on G-bundles on Σ^0 with reg. sing.s $\}/$isom

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Poisson scheme (∞-type)

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ $\Longrightarrow$ $\mathcal{M}_B$

$$\mathcal{M}_B \underset{\text{RHB}}{\cong} \mathcal{M}_{DR}$$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

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Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

Poisson variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \ldots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \ldots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

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$Q_i \in \underbrace{\mathfrak{t}_i}_{\text{cartan subalg.}} \subset \mathfrak{g}((z_i))$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Poisson variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \dots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

$\parallel\!\wr$ RHB

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$\nabla \cong dQ_i + \Lambda_i \frac{dz_i}{z_i} + \text{holom.}$

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Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

Wild Riemann surface $(\Sigma, \underset{\sim}{a}, \underset{\sim}{Q}) \implies$ wild character variety

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Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$

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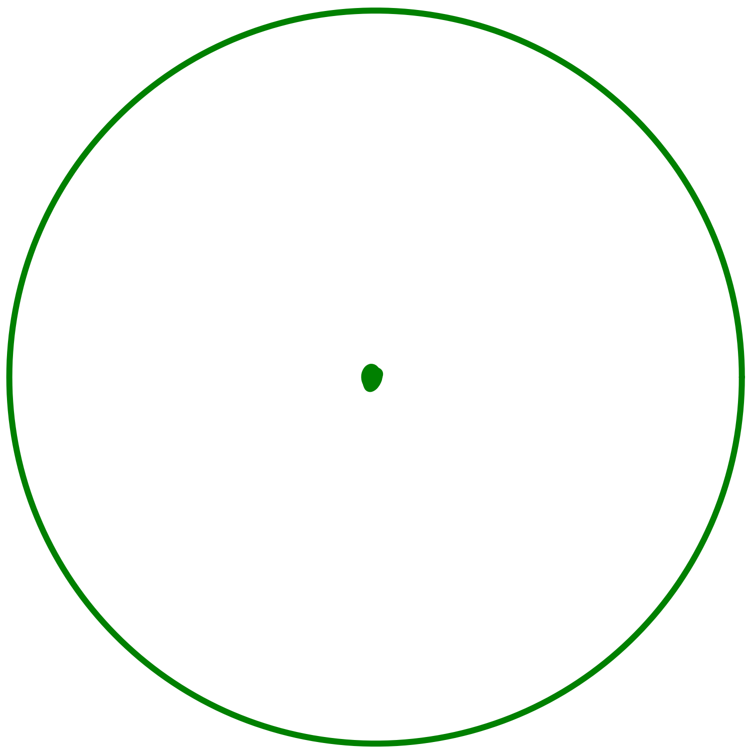

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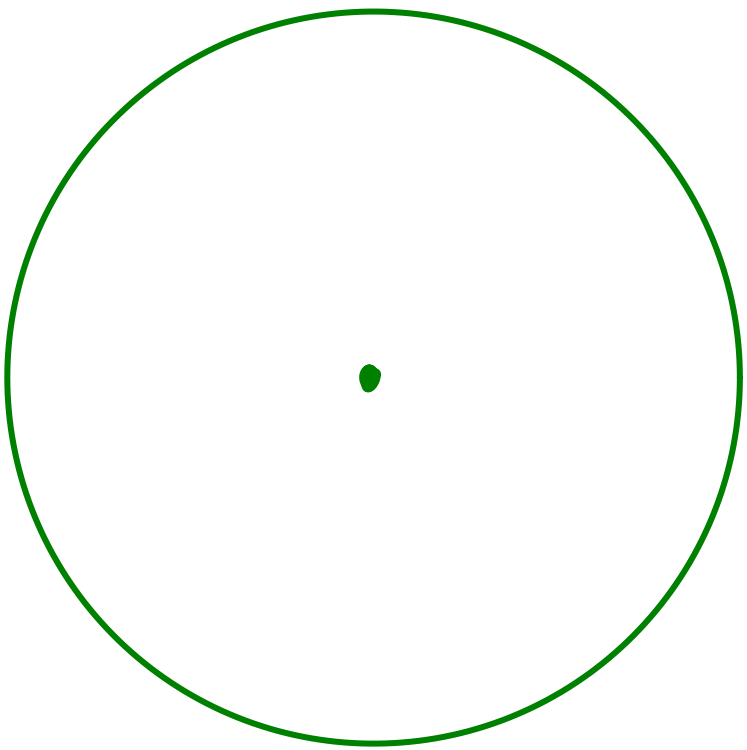

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formal fundamental solution:

$$\Phi = F(z)\, z^{\Lambda} e^{Q}$$

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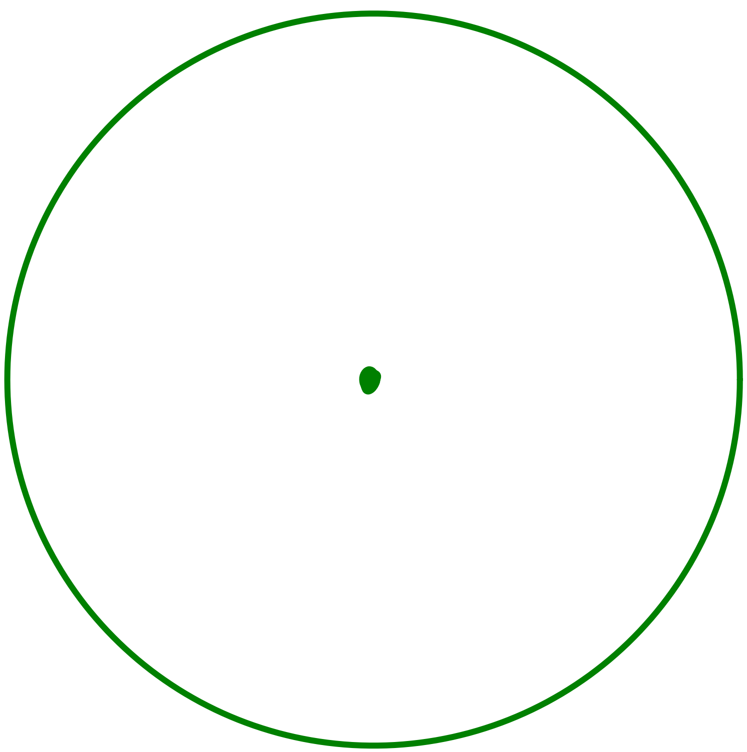

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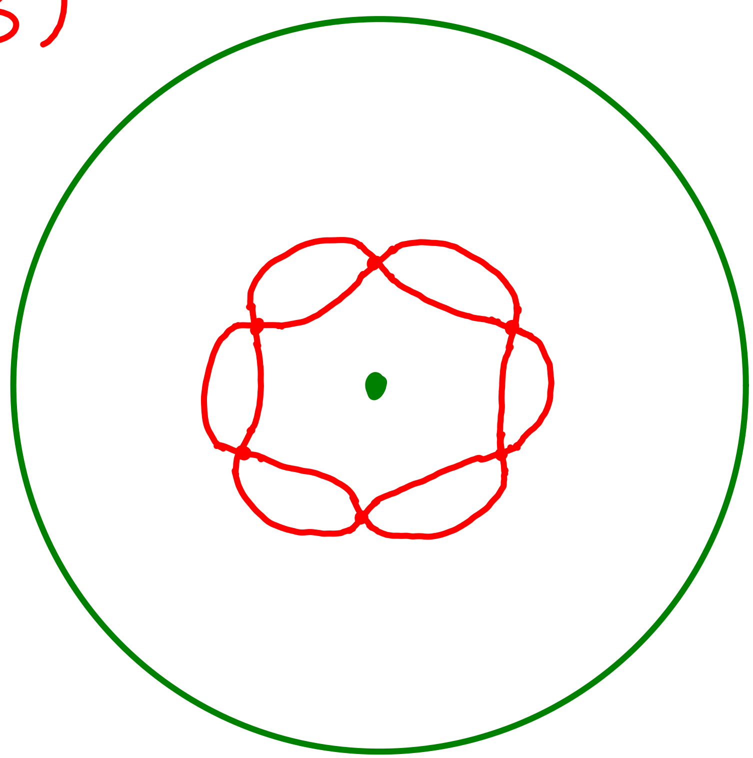

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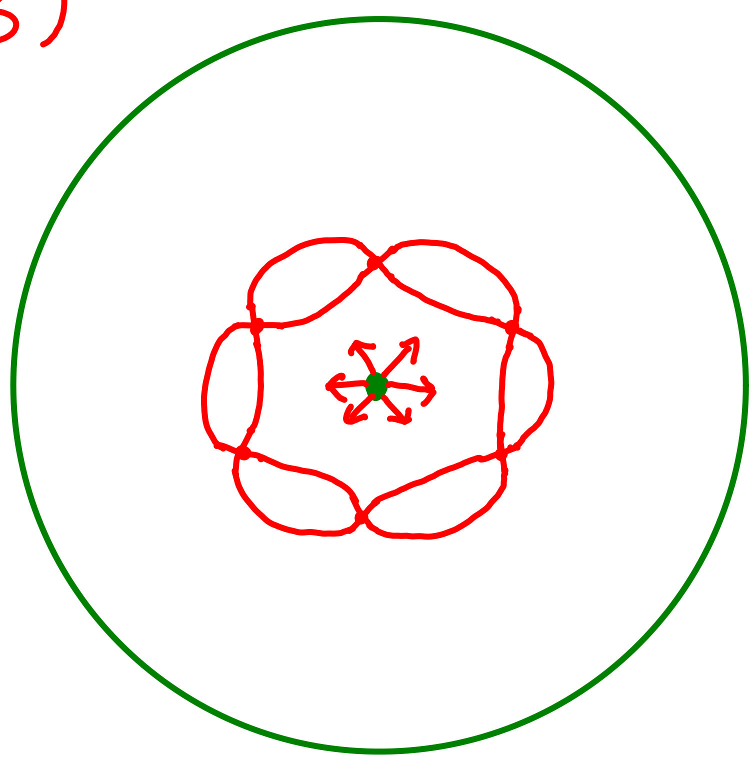

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- singular directions $\mathbb{A}$

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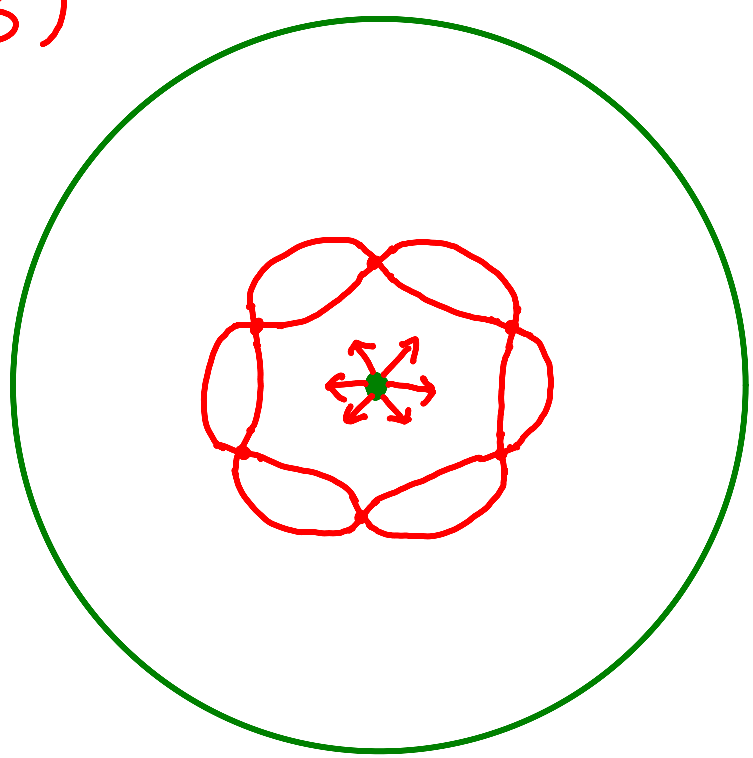

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$\exists$ "sum" F_i of F on each sector, i.e. isomorphism $\nabla_0 = dQ + \Lambda \frac{dz}{z} \cong \nabla$

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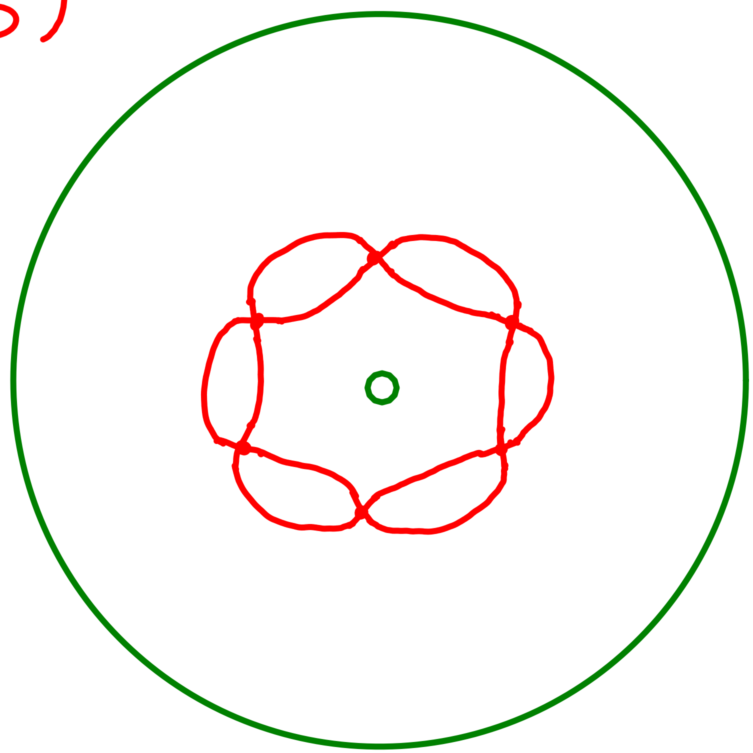

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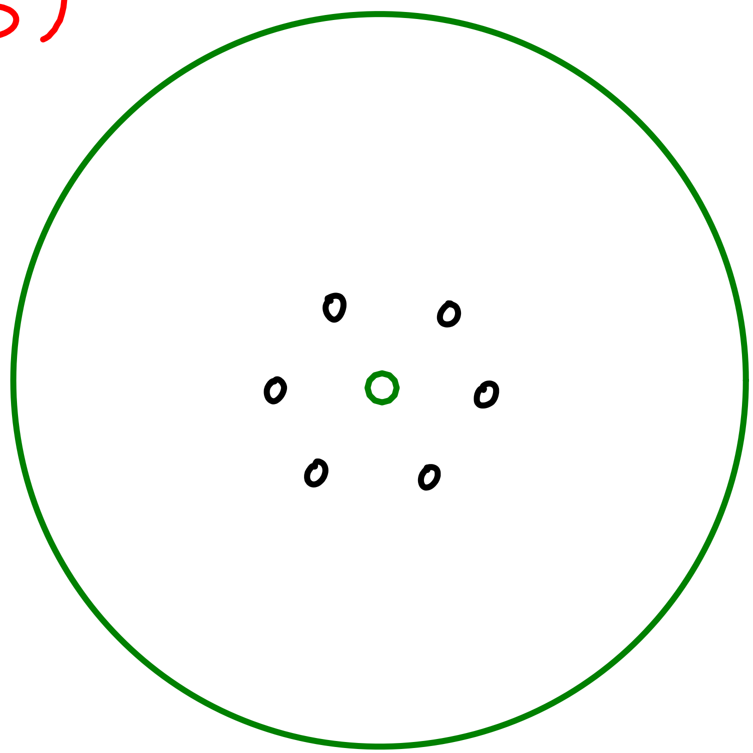

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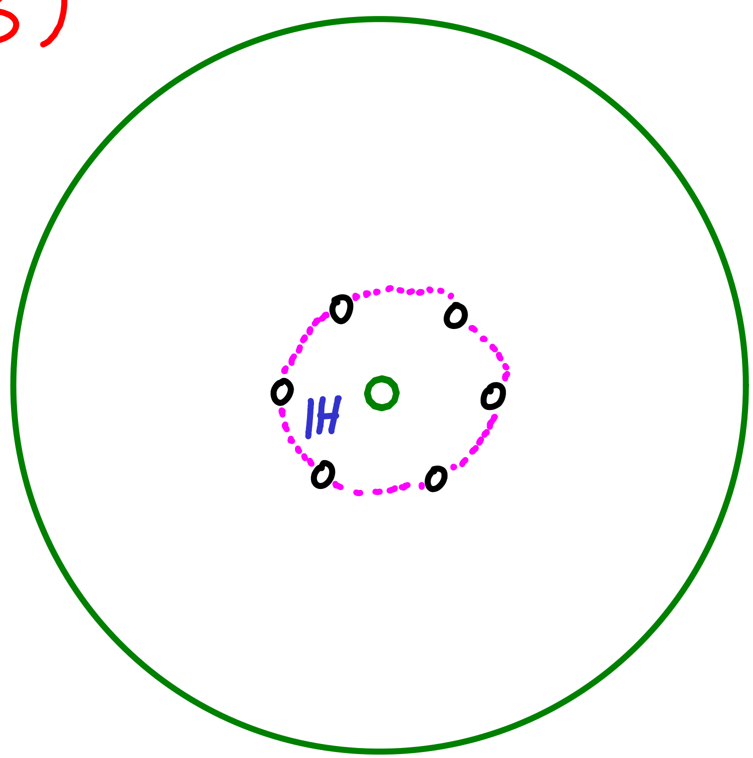

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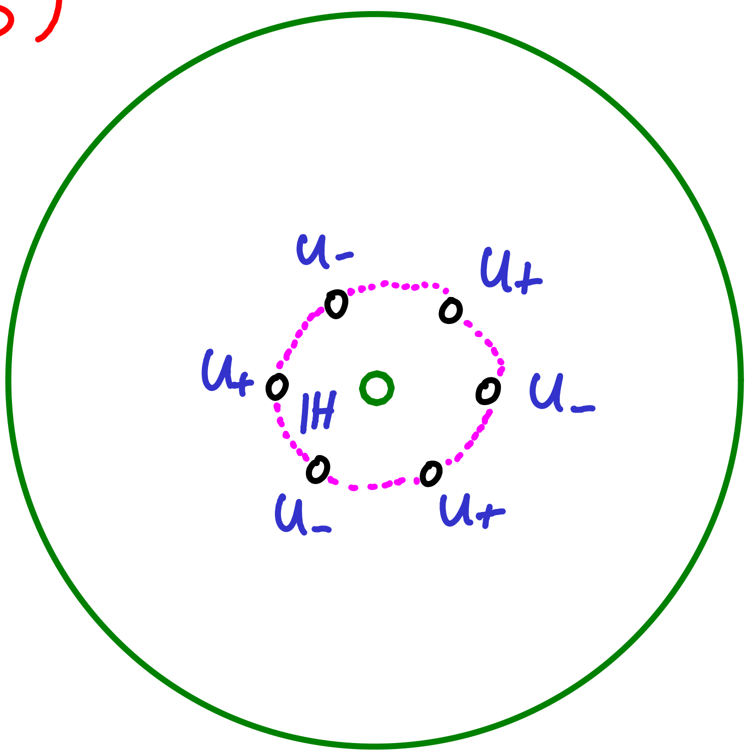

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- Stokes groups $\mathrm{Sto}_d \subset G$ $\forall d \in \mathbb{A}$

$\cong U_+$ or U_- here

$$\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}$$

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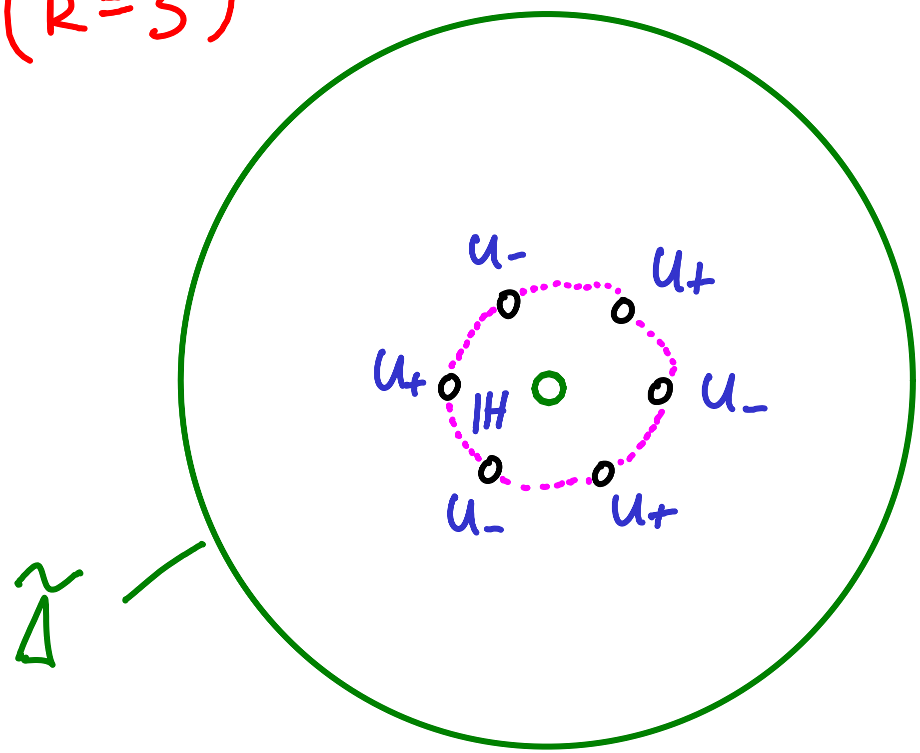

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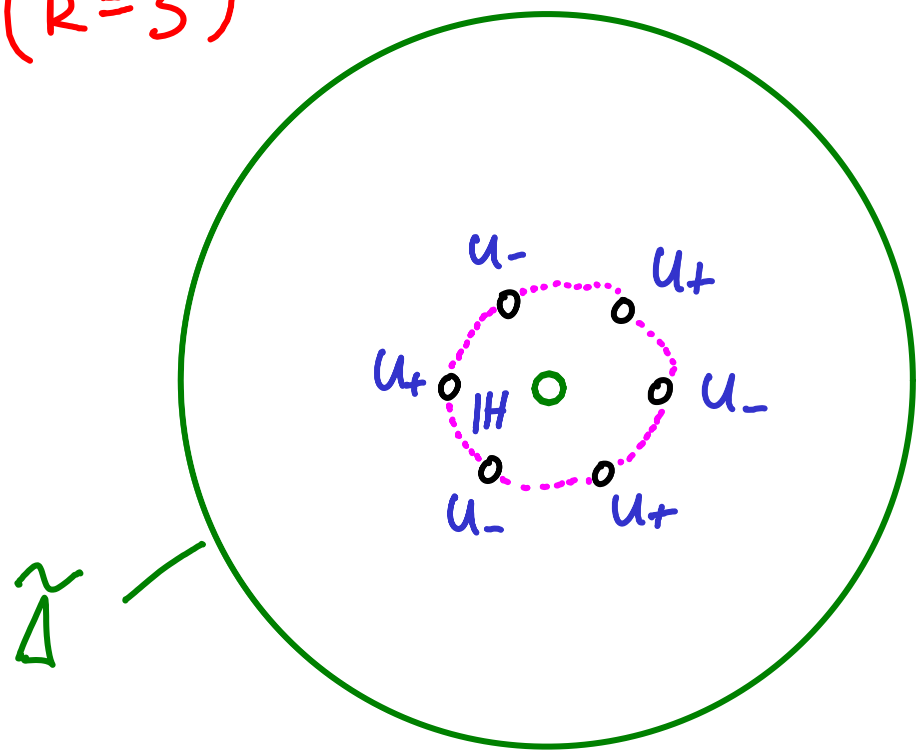

$$\nabla = dQ + \Lambda \frac{dz}{z} + \text{holom.}$$

Stokes local system:

- singular directions $/A$

$\exists$ "sum" F_i of F on each sector, i.e. isomorphism $\nabla_0 = dQ + \Lambda\frac{dz}{z} \cong \nabla$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

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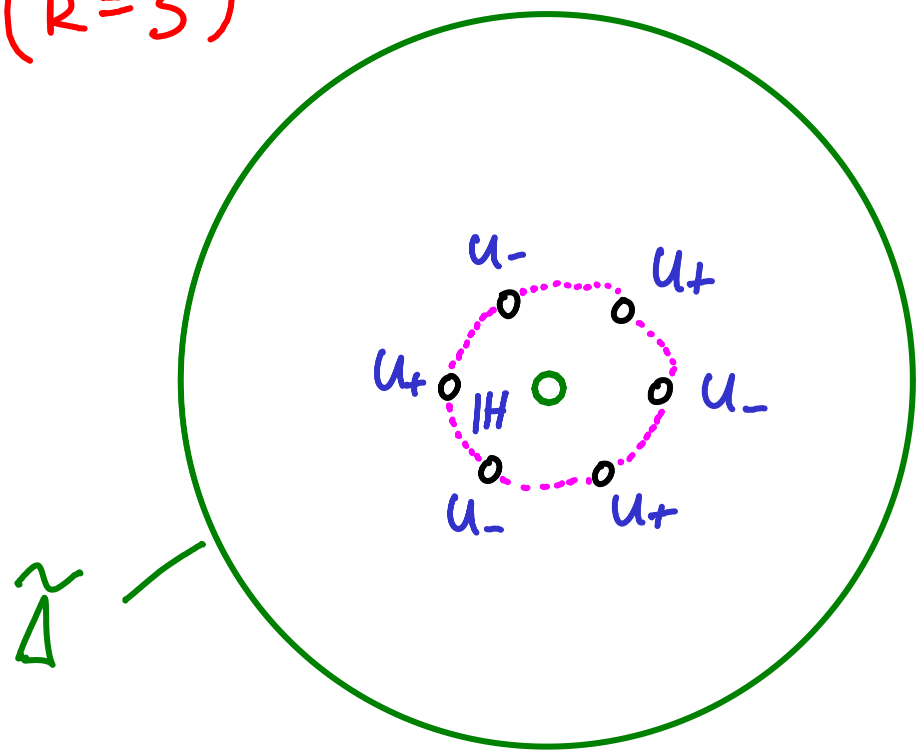

$\nabla = dQ + \Lambda \frac{dz}{z} + \text{holom.}$

Stokes local system:

G-local system on $\tilde{\Delta}$

- flat reduction to $H = C_G(Q)$ in $\mathbb{H}$
- monodromy around $e(d)$ in $\mathbb{S}to_d$ $\forall d \in \mathbb{A}$

- singular directions $\mathbb{A}$

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Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

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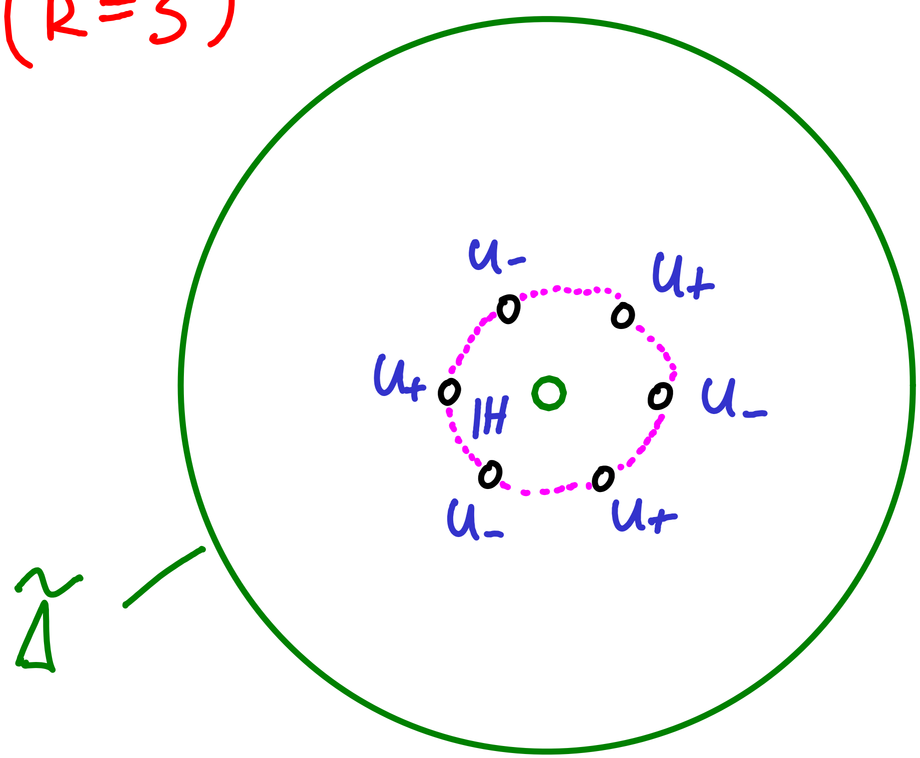

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Stokes local system:

G-local system on $\tilde{\Delta}$

- flat reduction to $H = C_G(Q)$ in $\mathbb{H}$
- monodromy around $e(d)$ in Sto_d $\forall d \in \mathbb{A}$

$$\{\text{connections with irreg. type } Q\} \cong \{\text{Stokes local systems on } \tilde{\Delta}\}$$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

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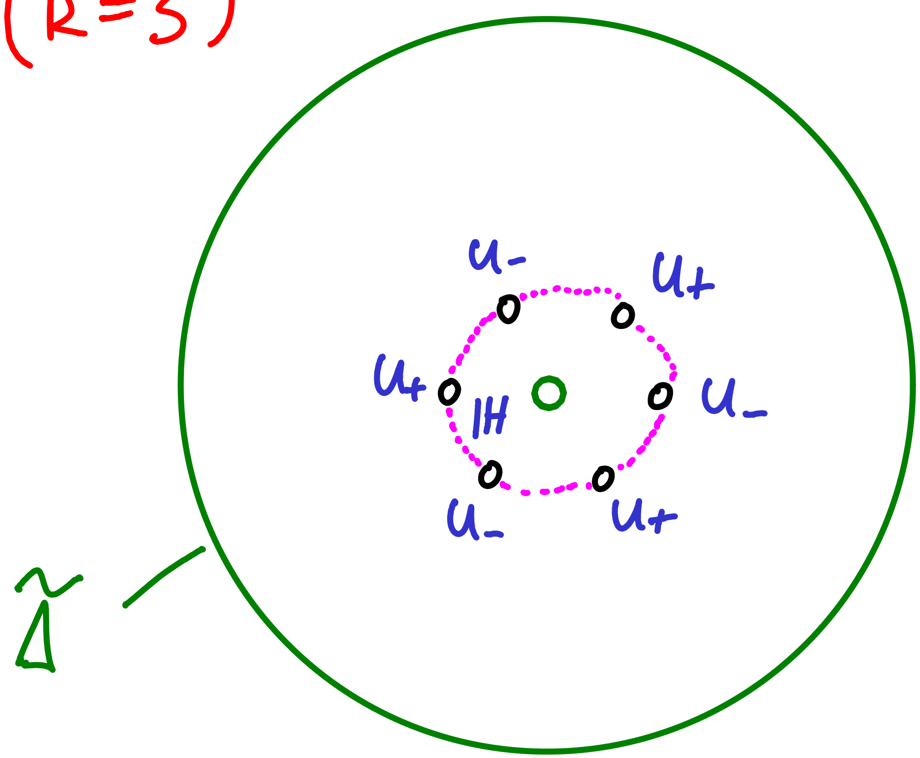

Classify?

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Wild Character Varieties

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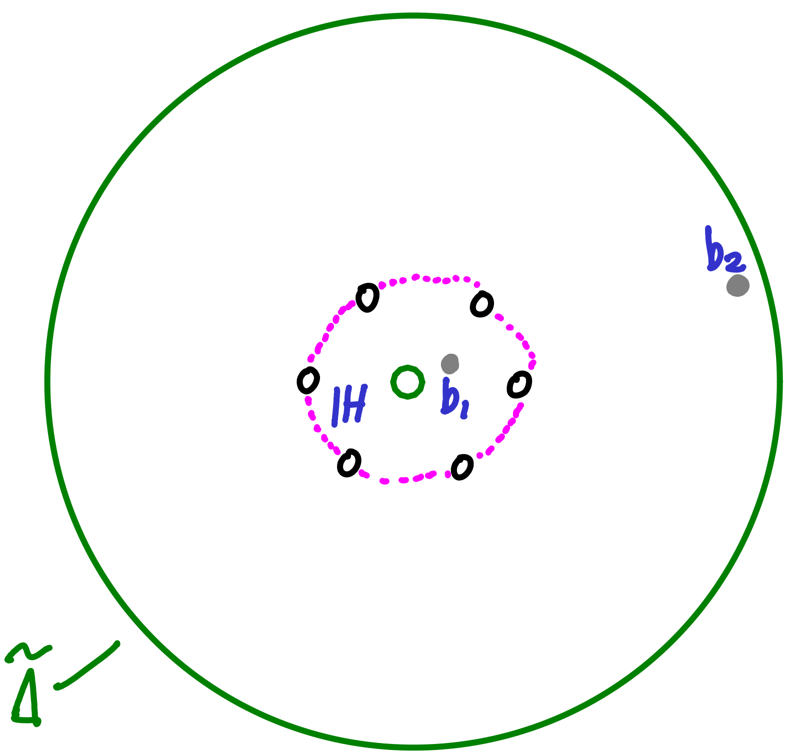

basepoints b_1, b_2

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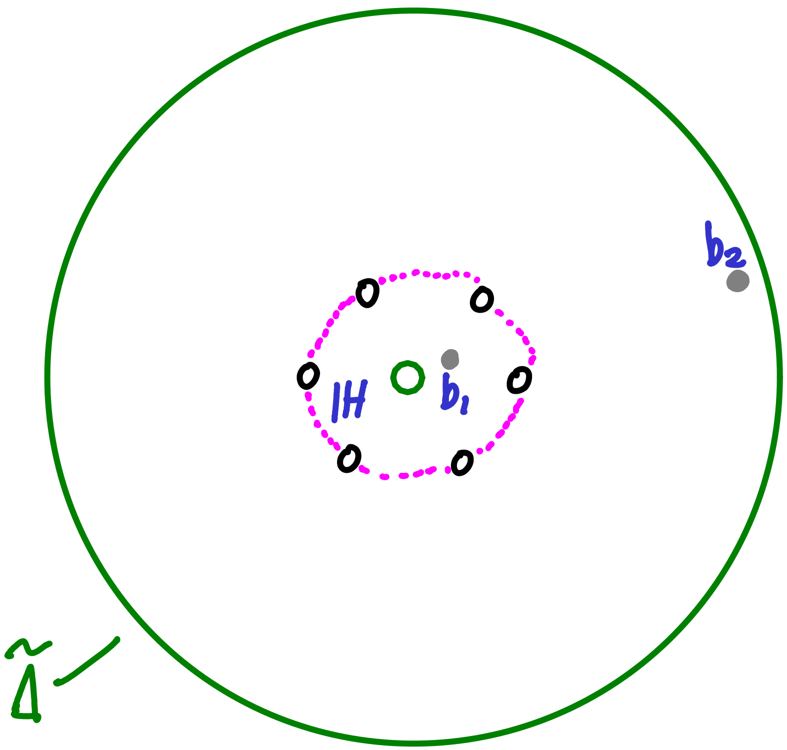

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$\Pi = \Pi_1(\tilde{\Delta}, \{b_1, b_2\})$

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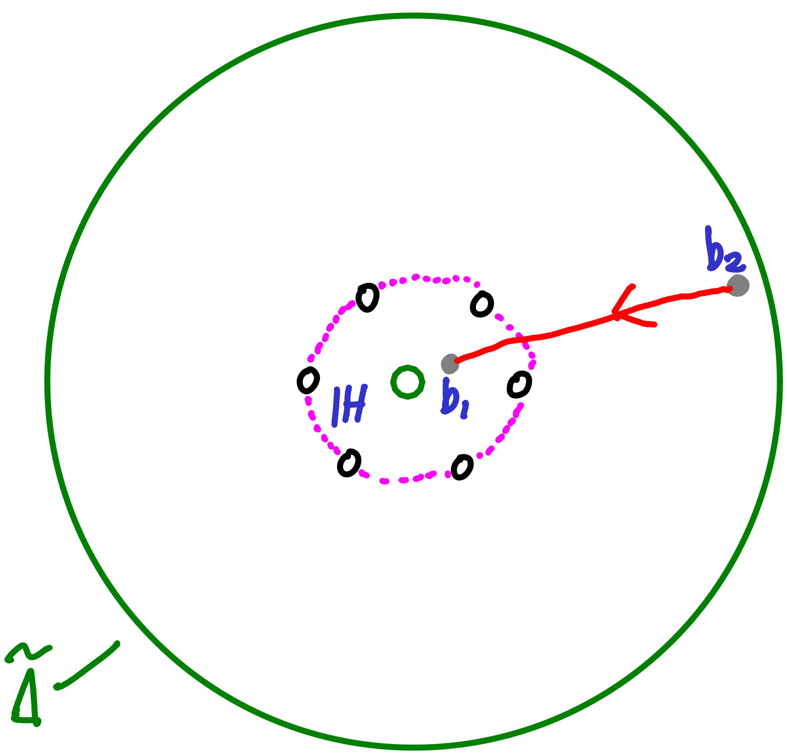

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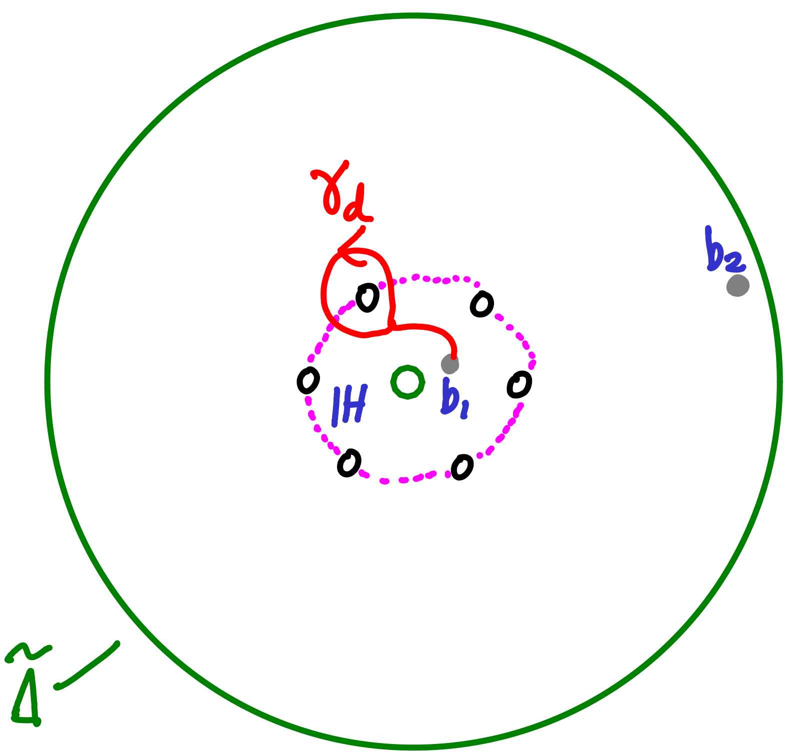

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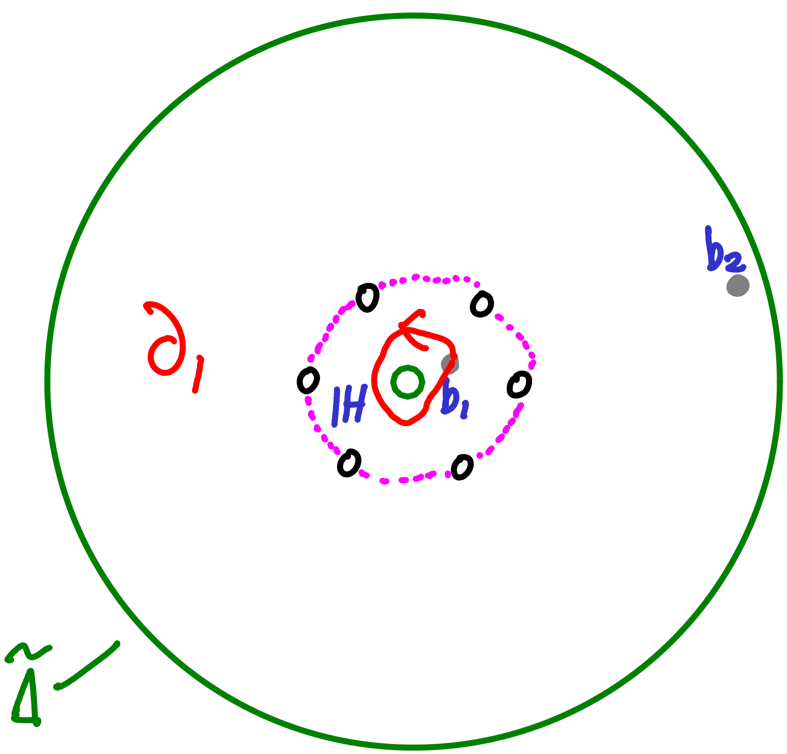

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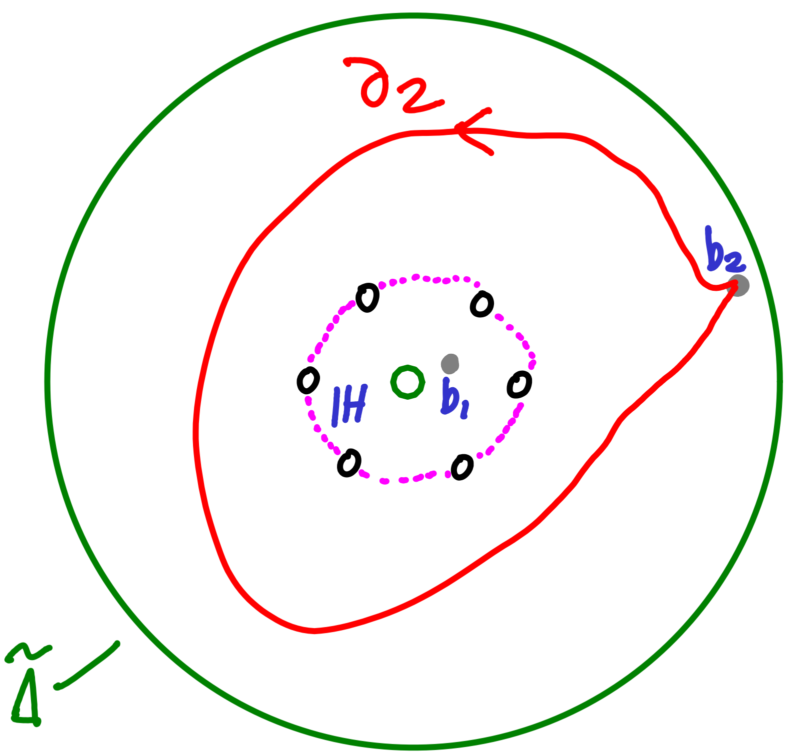

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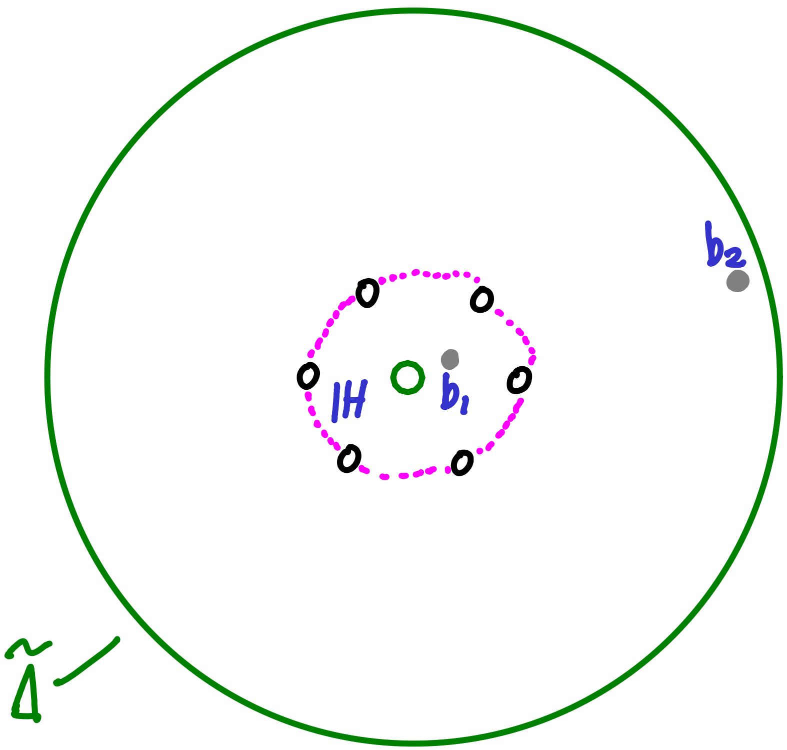

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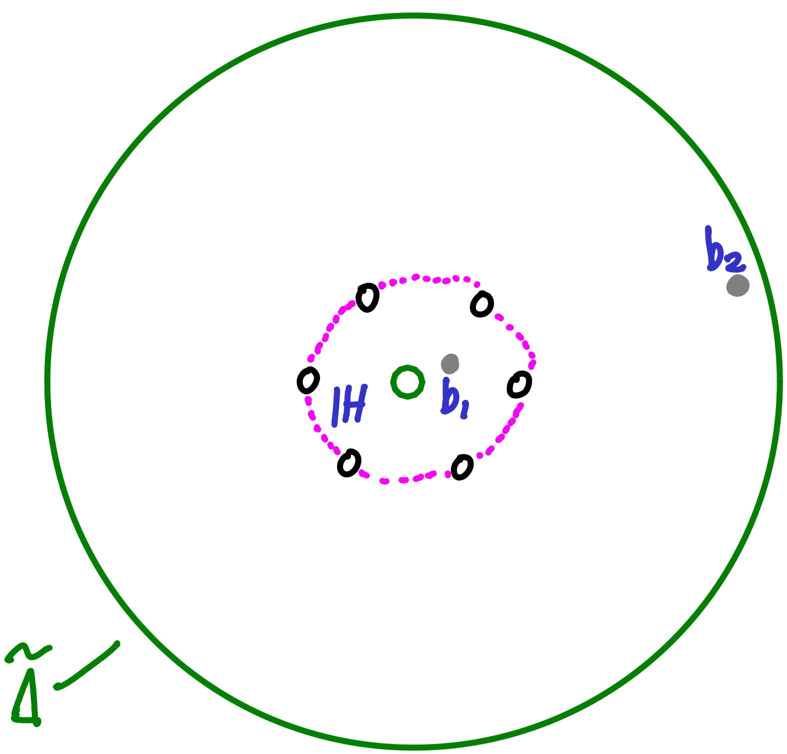

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$$\tilde{\mathcal{M}}_B = \mathrm{Hom}_g(\Pi, G)$$

$$= \left\{ \rho : \Pi \to G \;\middle|\; \begin{array}{l} \rho(\partial_1) \in H \\ \rho(\gamma_d) \in \mathrm{Sto}_d \quad \forall d \in \mathcal{A} \end{array} \right\}$$

Wild Character Varieties

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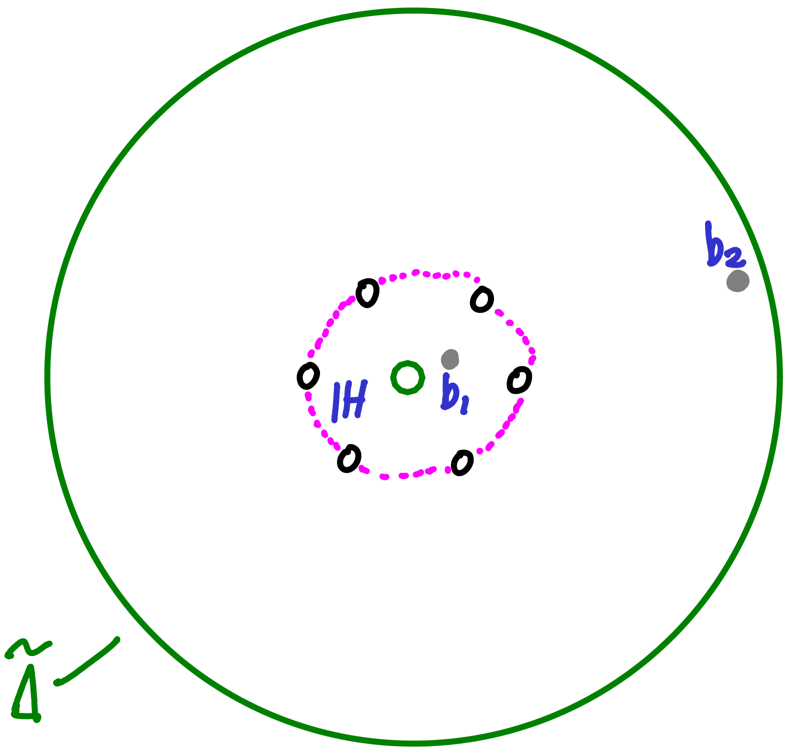

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Thm (arXiv 0203.*****)

$\tilde{\mathcal{M}}_B$ is a quasi-Hamiltonian $G \times H$ space

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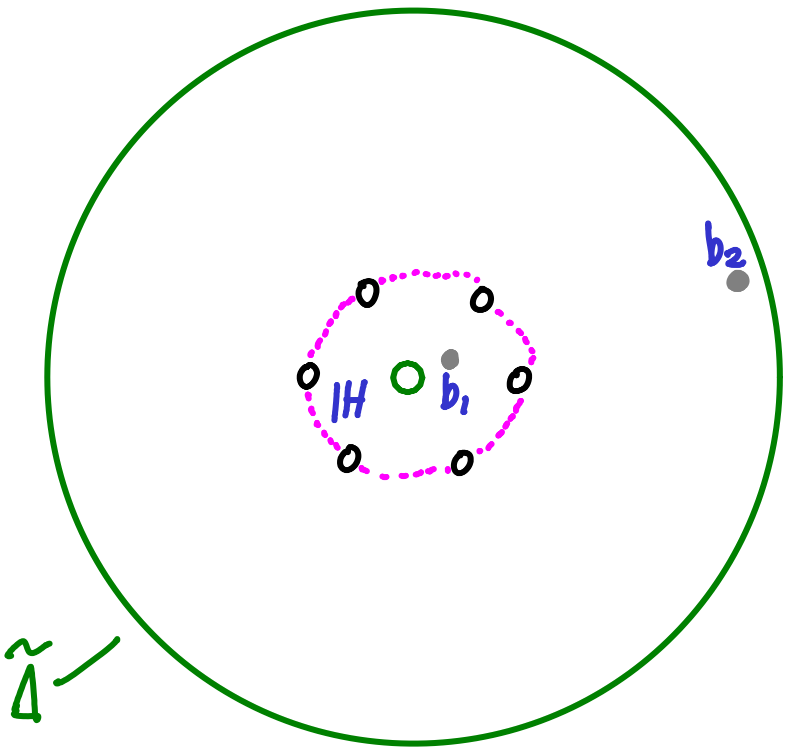

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$$\tilde{\mathcal{M}}_B = \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$$
$$\cong G \times (U_+ \times U_-)^k \times H$$

Thm (arXiv 0203.****)

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Thm (arXiv 0203.****)

$\mathcal{A}(Q) = G \times (U_+ \times U_-)^k \times H$ is a quasi-Hamiltonian $G \times H$ space ("fission space")

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$$(C, \underset{\sim}{S}, h) \qquad \underset{\sim}{S} = (S_1, \ldots, S_{2k}) \qquad S_{\text{odd/even}} \in U_{+/-}$$

Moment map $\quad \mu(C, \underset{\sim}{S}, h) = (C^{-1} h S_{2k} \cdots S_2 S_1 C, \; h^{-1}) \in G \times H$

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

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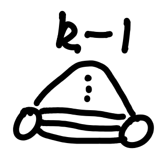
$\Gamma =$ 

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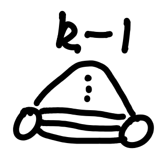
$\Gamma =$ 

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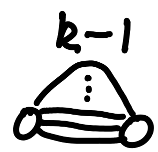
$=: \operatorname{Rep}^*(\Gamma, V)$ $\Gamma =$ 

, $V = \mathbb{C} \oplus \mathbb{C}$

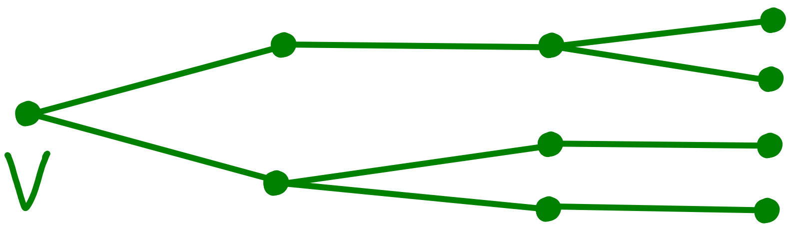
[Similarly for $V = V_1 \oplus V_2$ any dimension
(2009-2015) Γ any "fission graph"]

$$\mu(a_1, \dots, b_{k-1}) = \left((a_1, b_1, \dots, a_{k-1}, b_{k-1}), (b_{k-1}, \dots, b_1, a_1)^{-1}\right)$$

Fission graphs (arXiv 0806, apx C) $G = GL(V)$

$$Q = A_r/z^r + \cdots + A_1/z \qquad (A_i \in \mathfrak{t})$$

$$= A_r w^r + \cdots + A_1 w \qquad w = 1/z$$

$r = 3$: 

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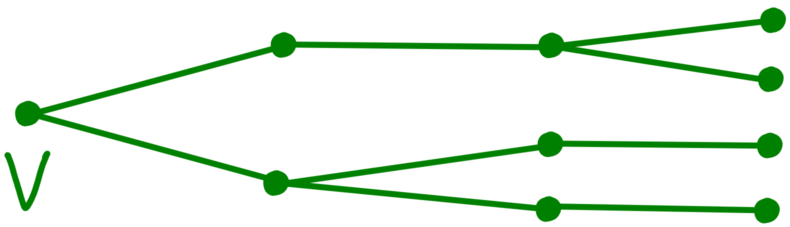
$r = 3$: 

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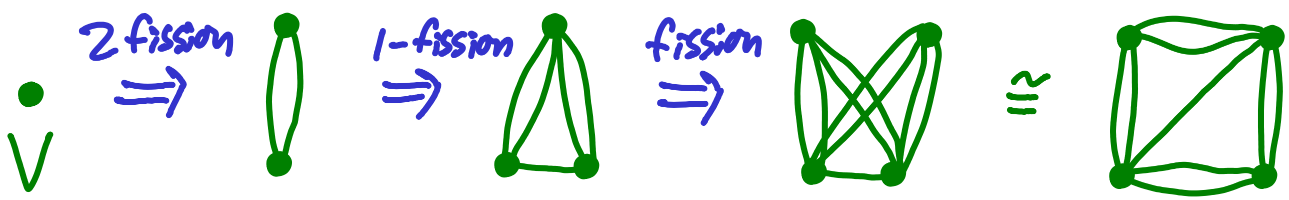
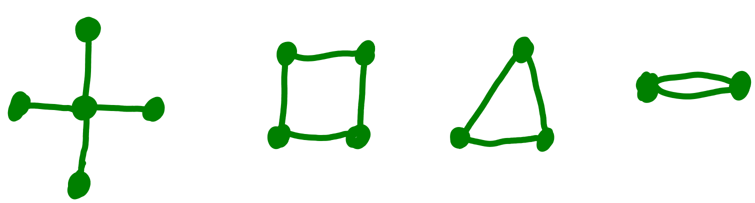
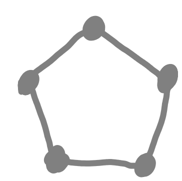


"fission graph" $\Gamma(Q)$

- $r=2$ get all complete k-partite graphs
- e.g.  but <u>not</u> 

$Q = \mathrm{diag}(q_1, \ldots, q_n) \Rightarrow$ nodes $= \{1, \ldots, n\}$, # edges $i \leftrightarrow j = \deg_w(q_i - q_j) - 1$

Wild Character Varieties

In this example $\left((\mathbb{P}^1, 0, Q) \quad Q = A/z^k, \quad GL_2(\mathbb{C}) \right)$

$$\mathcal{M}_B = \tilde{\mathcal{M}}_B /\!\!/_{(q_1,q_2)} H$$

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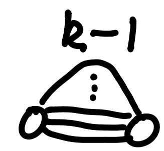
$\Gamma =$ 

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E.g. $k=3$ (Painlevé 2 Betti space)

$$\mathcal{M}_B \cong \left\{ xyz + x + y + z = b - b^{-1} \right\} \quad b \in \mathbb{C}^* \text{ constant}$$

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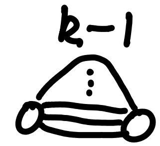
$\Gamma =$ 

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Also $\mathcal{M}^* \cong \mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} H$ "Nakajima/additive quiver variety"

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$$\begin{array}{ccc} \mathcal{M}^* & \overset{RHB}{\Longrightarrow} & \mathcal{M}_B \\ \cong & & \cong \\ \mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} H & & \mathrm{Rep}^*(\Gamma, V) /\!\!/_{q} H \end{array}$$

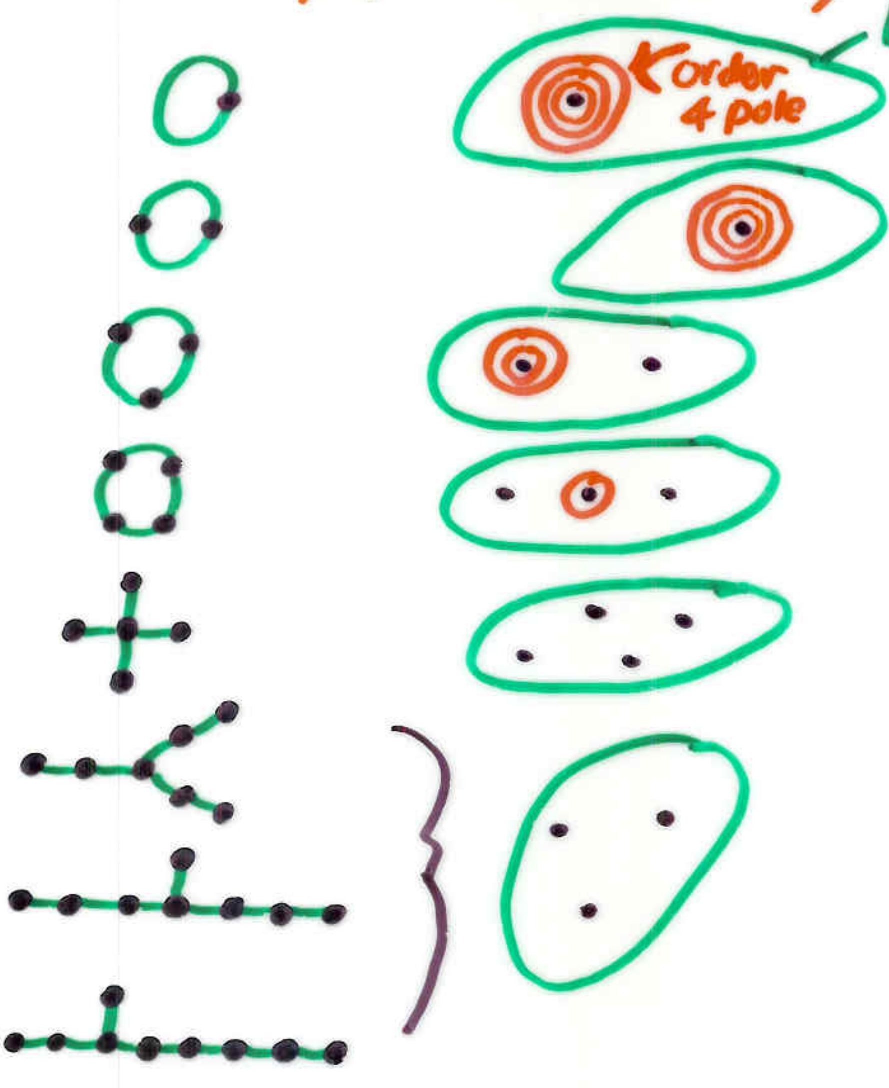
Examples

"Quaternionic Curves" $\begin{cases} \dim_{\mathbb{H}} = 1 \\ \text{real 4 mfds} \end{cases}$

– simplest examples have open pieces

$\cong$ Kronheimer's ALE spaces

{ Cx. sympl. not isometrically



A_0	GL_2 – not generic
A_1	GL_2
A_2	GL_2
A_3	GL_2
D_4	GL_2
E_6	GL_3
E_7	GL_4 } not generic
E_8	GL_6 } not generic

cf. Cherkis-Kapustin, ..., Painlevé...

[Slide from talk at IAS Princeton, November 2007]

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607)

affine Weyl group	minimal rank of bundles	pole orders
E_8	6	1+1+1
E_7	4	1+1+1
E_6	3	1+1+1
D_4	2	1+1+1+1
$A_3 = D_3$	2	2+1+1
D_2	2	2+2
D_1	2	2+2
D_0	2	2+2
A_2	2	3+1
A_1	2	4
A_0	2	4

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Tame ← | → Wild

	Tame				Wild						
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minimal rank of bundles	6	4	3	2	2	2	2	2	2	2	2
pole orders	1+1+1	1+1+1	1+1+1	1+1+1+1	2+1+1	2+2	2+2	2+2	3+1	4	4

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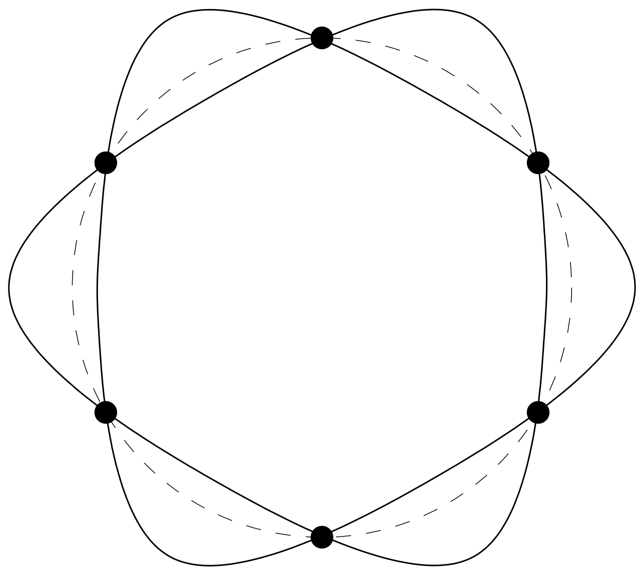
E_8 E_7 E_6

P_6	P_5	P_3	P_3'	P_3''
D_4	$A_3 = D_3$	D_2	D_1	D_0
		A_2	A_1	A_0
		P_4	P_2	P_1

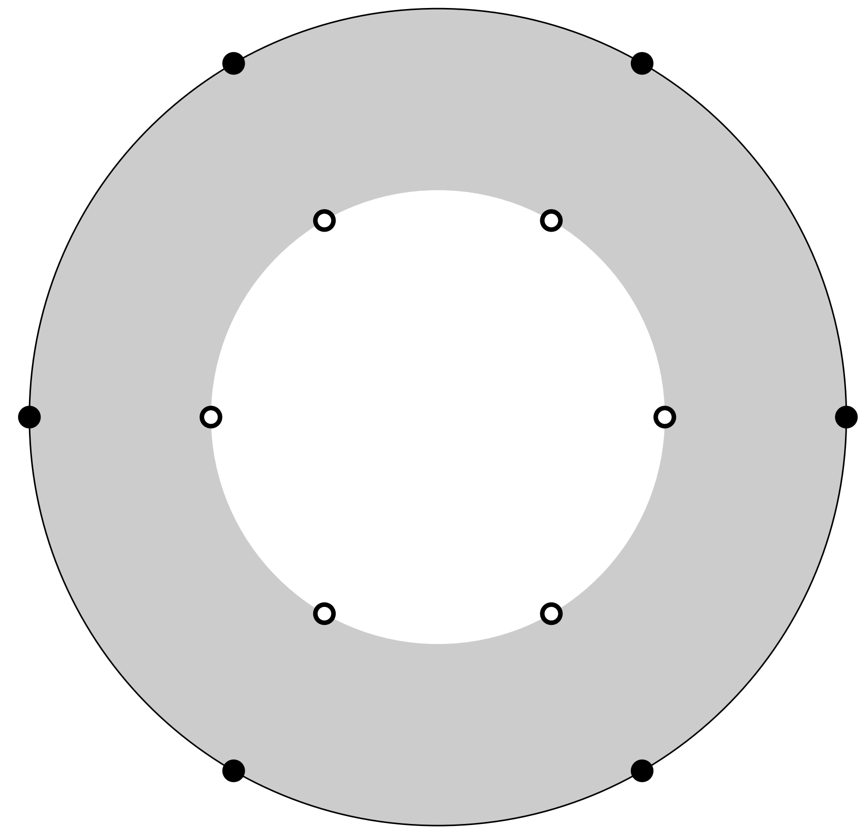
Phase spaces for Painlevé differential equations

Stokes structures

(Sibuya 1975, Deligne 1978, Malgrange 1980 ...)



Stokes diagram with Stokes directions



Halo at ∞ with singular directions

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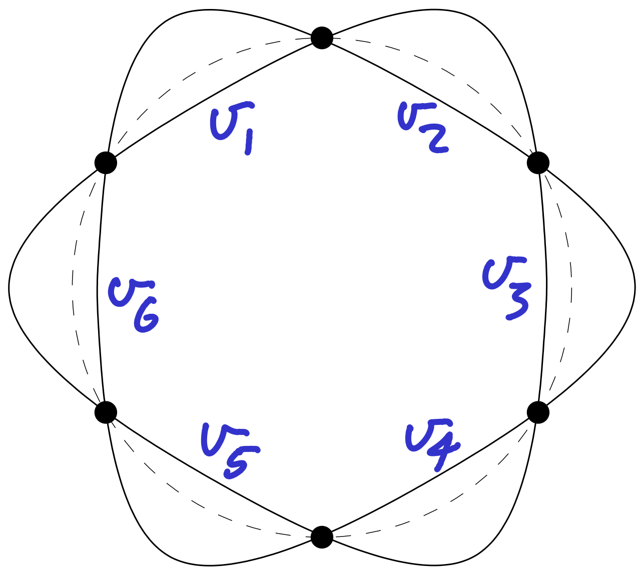

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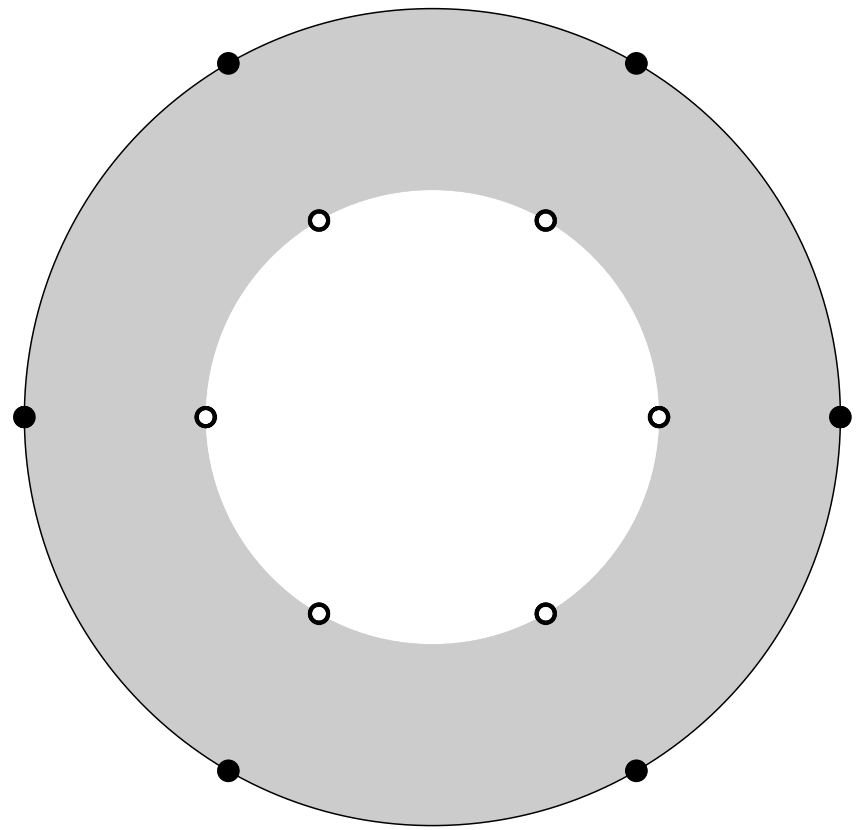

Halo at ∞ with singular directions

Subdominant solutions $\quad v_i \not\parallel v_{i+1}$

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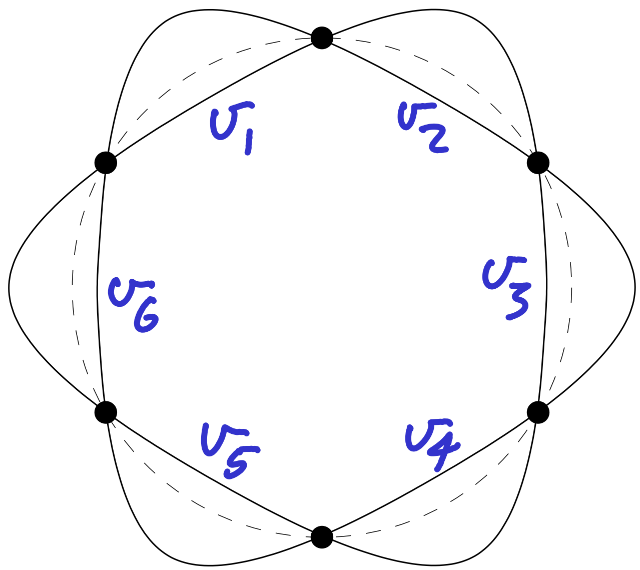

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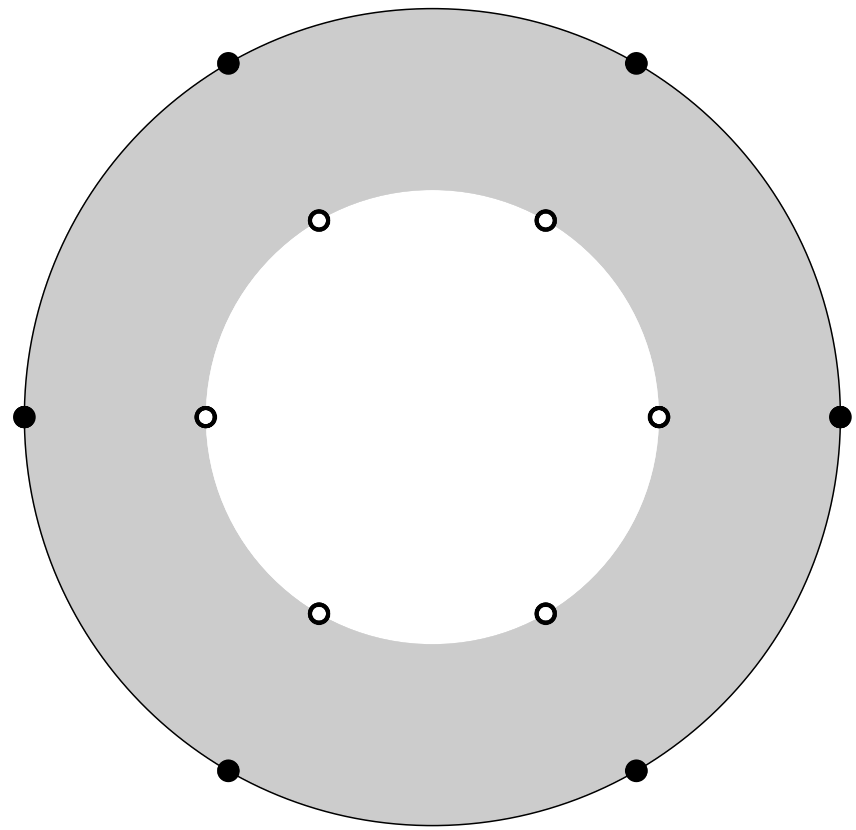

Halo at ∞ with singular directions

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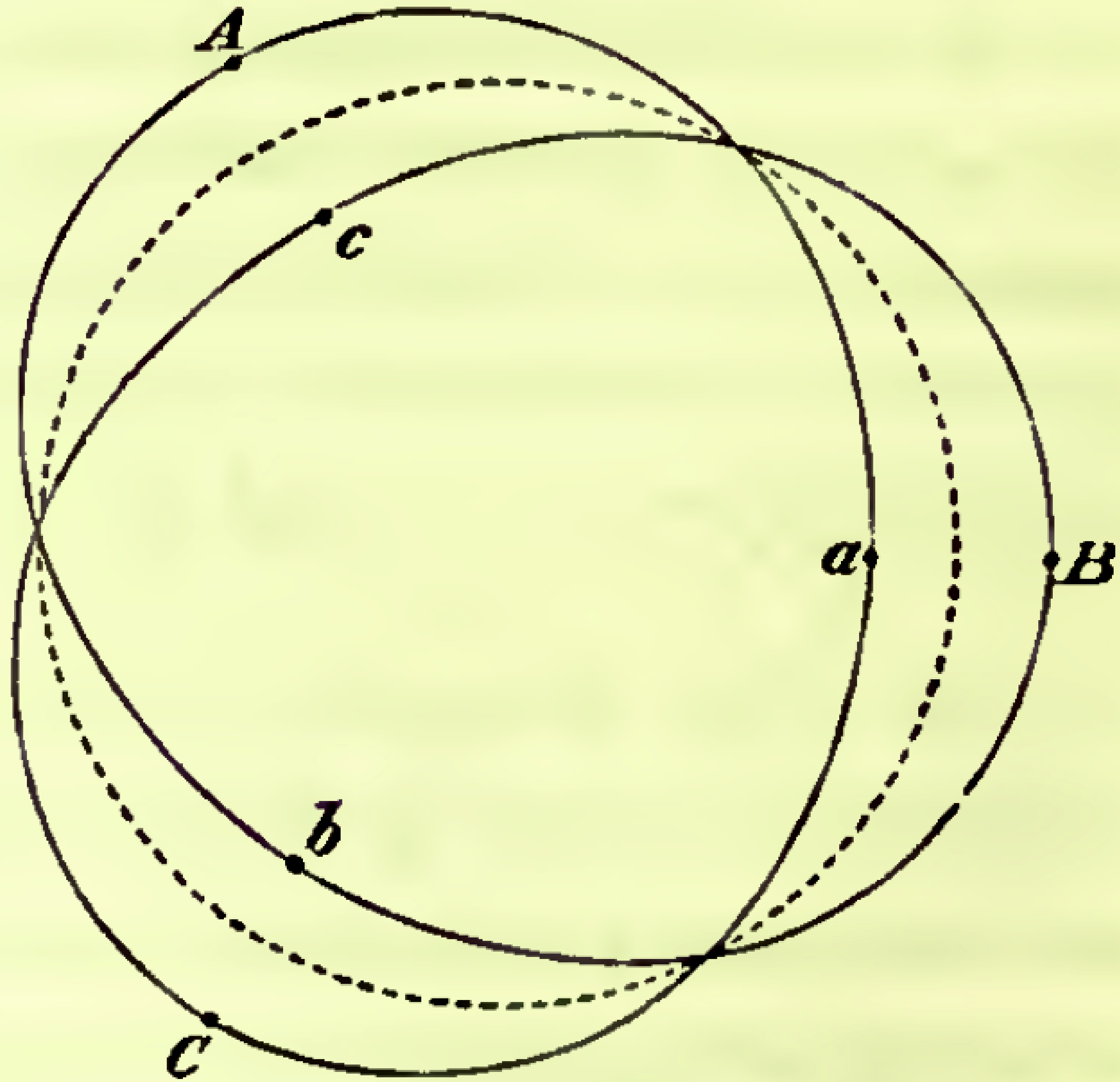
$$\mathcal{M}_B \cong \{ xyz + x + y + z = b - b^{-1} \}$$

$$\cong \left\{ (p_1, \dots, p_6) \in (\mathbb{P}^1)^6 \;\middle|\; \begin{array}{c} p_i \neq p_{i+1} \bmod 6 \\ \dfrac{(p_1 - p_2)(p_3 - p_4)(p_5 - p_6)}{(p_2 - p_3)(p_4 - p_5)(p_6 - p_1)} = b^2 \end{array} \right\} \Big/ PSL_2(\mathbb{C})$$

other the *inferior*
the existence and
different values of
ng a radius vector
e angle θ take two
and inwards from
al to the real part
perior and inferior
or in other words
nvenience suppose
d with the radius.

Fig. 1.



The curve will evidently have the form represented

§3 Odd continuants (work with D. Yamakawa)

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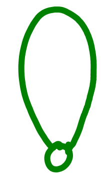
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$  $A_0^{(1)}$

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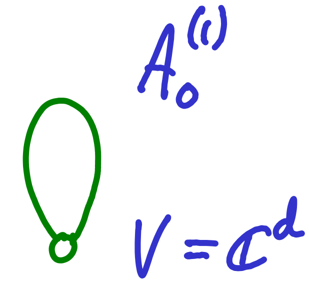
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$ 

- In additive case get $\tilde{\mathcal{M}}^* \cong T^*\mathrm{End}(V)$, $\mu = AB - BA$

  (PB 2008, unpublished)

  - So get Calogero-Moser spaces, ADHM spaces as $\mathcal{M}^*$
  - $\mathcal{M}^* = \mathbb{C}^2$ for Painlevé 1 $(d=1)$

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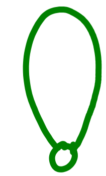
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$ 

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- **Thm** $\mathrm{Rep}^*(\Gamma, V) := \{ a,b,c \in \mathrm{End}(V) \mid (a,b,c) = 1 \}$

  is a quasi-Hamiltonian $GL(V)$-space of dimension $2d^2$

  with moment map $\mu(a,b,c) = (c,b,a)$

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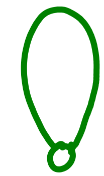
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$  $A_0^{(1)}$ $V = \mathbb{C}^d$

- In additive case get $\widetilde{\mathcal{M}}^* \cong T^*\mathrm{End}(V)$, $\mu = AB - BA$
  (PB 2008, unpublished)

  – So get Calogero–Moser spaces, ADHM spaces as $\mathcal{M}^*$
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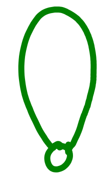
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$  $A_0^{(1)}$, $V = \mathbb{C}^d$

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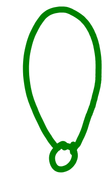
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$  $A_0^{(1)}$ $V = \mathbb{C}^d$

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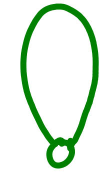
§3 Odd continuants (work with D. Yamakawa) $\Gamma =$  $V = \mathbb{C}^d$

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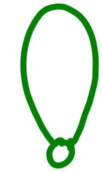
$\Gamma =$  $V = \mathbb{C}^d$

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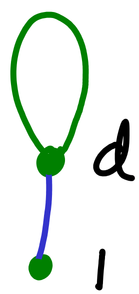
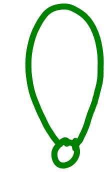
Other reductions:

$\mathrm{Rep}^*\Big($ 

$, \mathbb{C}^d \oplus \mathbb{C} \Big) /\!\!/_q H \;\cong\; \mathcal{M}_B\big(hP_1^{(d)}\big)$ dim $2d$

higher/hyperbolic/Hilbert Painlevé 1

§3 Odd continuants (work with D. Yamakawa) $\Gamma =$  $V=\mathbb{C}^d$

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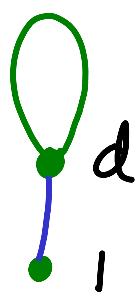
$\mathrm{Rep}^*\Big($ 

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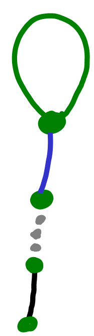
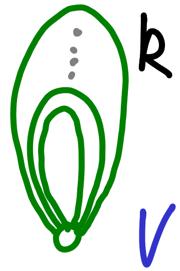
$\mathrm{Rep}^*\Big($  $, \oplus V_i \Big) /\!/_q H \cong \mathcal{M}_B(\text{matrix } P_1)$

§3 Odd continuants (work with D. Yamakawa)

More generally if $\Gamma =$ 

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$(r = 2k+1)$

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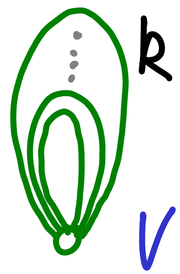
More generally if $\Gamma =$ 

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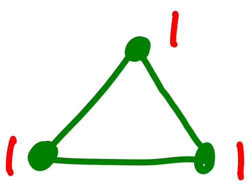
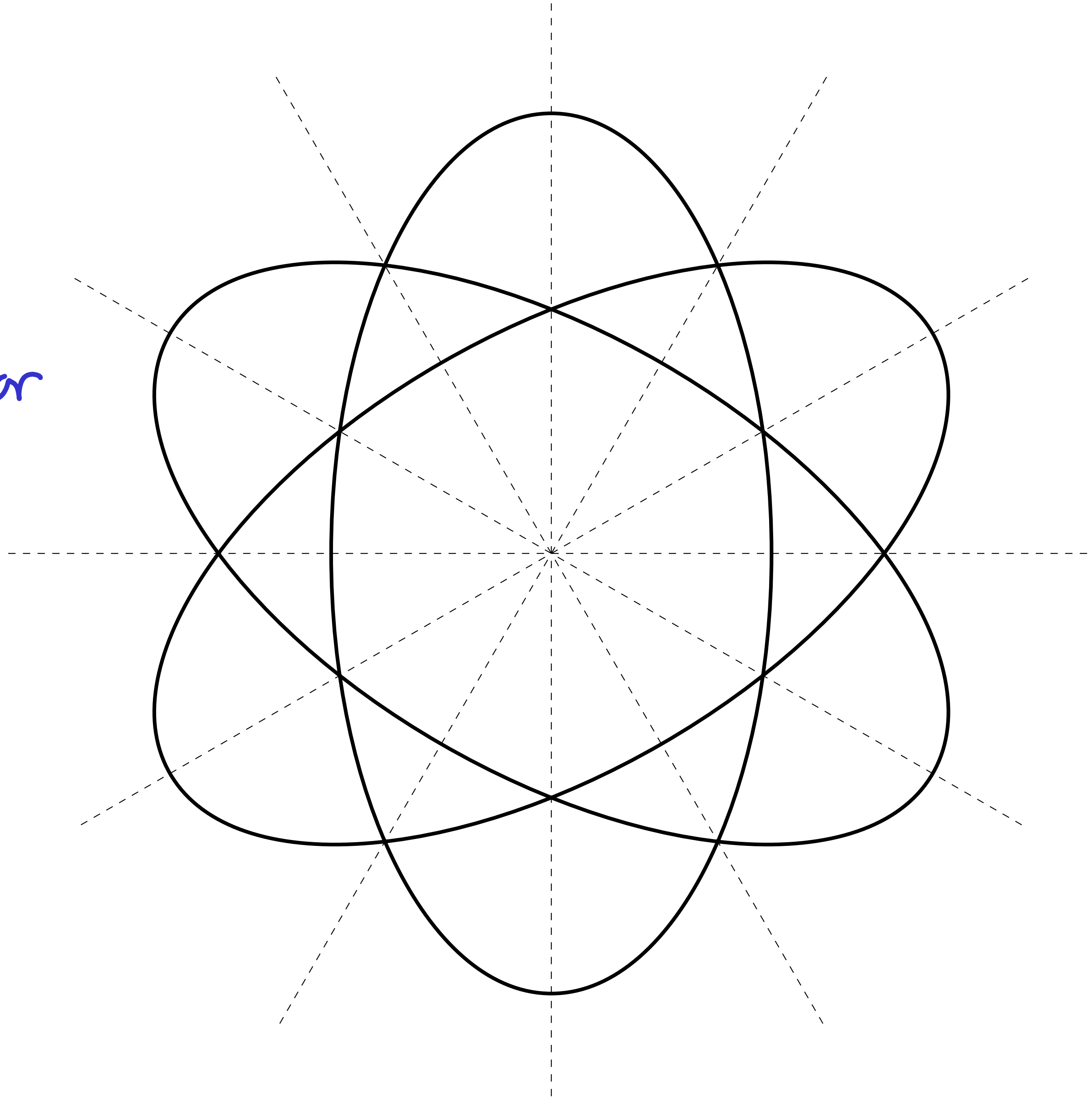
– and similarly for any twisted irregular type Q (any G)

Stokes diagram for

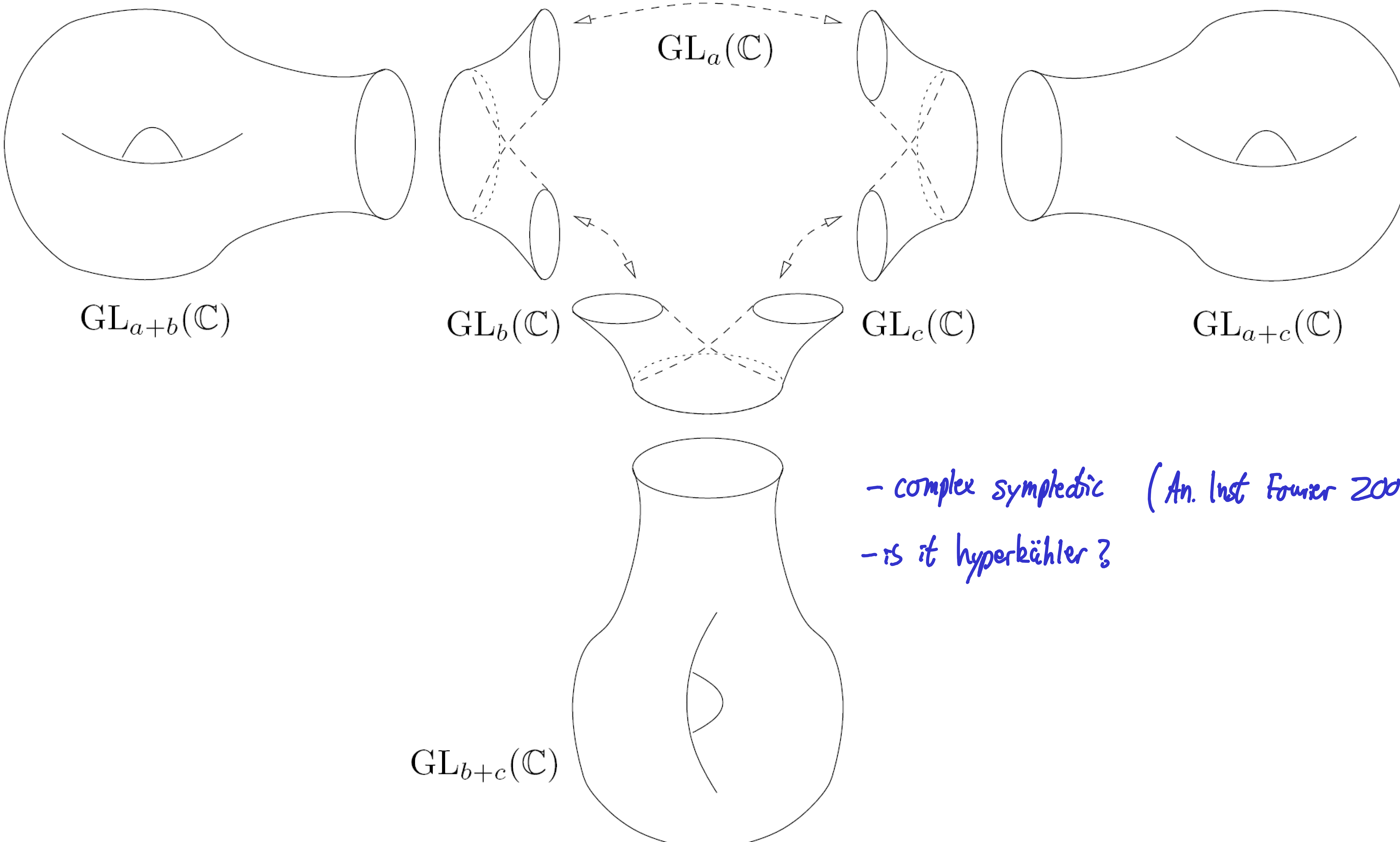


(generic reading)



LMS

150 YEARS



- complex symplectic (An. Inst Fourier 2009)
- is it hyperkähler ?