First steps in global Lie theory: wild Riemann surfaces, their character varieties and topological symplectic structures

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- see also short survey arxiv: 1703 for more background
- course notes: ~/cours23/

Geometrically, what are the six Painlevé equations* trying to tell us?

* Picard, Painlevé, R. Fuchs, Gambier

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« Les Mathématiques constituent un continent solidement agencé, dont tous les pays sont bien reliés les uns aux autres ; l'œuvre de Paul Painlevé est une île originale et splendide dans l'océan voisin »

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Geometrically, what is the Riemann–Hilbert–Birkhoff correspondence* trying to tell us?

* Stokes, Birkhoff, Malgrange, Sibuya, Jurkat, Deligne, Écalle, Martinet, Ramis, ...

$G = GL_n(\mathbb{C})$ (or any other complex reductive group)

Riemann surface Σ $\rightsquigarrow$ character variety

$$\mathcal{M}_B = \mathcal{R}/G$$

$$\mathcal{R} = \mathrm{Hom}(\pi_1(\Sigma, b), G)$$

representation variety

wild Riemann surface $\underset{\sim}{\Sigma}$ $\rightsquigarrow$ wild character variety

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wild representation variety

Thm (B.-Yamakawa)

$\mathcal{M}_B$ is alg. Poisson variety, points are the reductive Stokes representations,

any admissible deformation of $\underset{\sim}{\Sigma}$ $\Rightarrow$ local system of Poisson varieties

Lie theory

$$\mathfrak{g} \longrightarrow G$$

$$X \longmapsto \exp(X)$$

Connection $\quad \dfrac{X}{2\pi i} \dfrac{dz}{z} \longmapsto$ monodromy

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moduli spaces: $\quad \mathcal{M}^* \longrightarrow \mathcal{M}_B \quad$ wild character variety

(same dimension)

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moduli spaces: $\quad \mathcal{M}^* \hookrightarrow \mathcal{M}_{DR} \xrightarrow{\cong} \mathcal{M}_B$ wild character variety (same dimension)

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moduli spaces:

$$\begin{tikzcd} \mathcal{M}^* \arrow[r, hook] & \mathcal{M}_{DR} \arrow[r, "\cong"] \arrow[dr, "\cong"] & \mathcal{M}_B \\ & & \mathfrak{M} \end{tikzcd}$$

$\mathcal{M}_B$: wild character variety (same dimension)

$\mathfrak{M}$: wild harmonic bundles (2d self-duality)

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moduli spaces:

$$\begin{tikzcd}
\mathcal{M}^* \arrow[r, hook] & \mathcal{M}_{DR} \arrow[r, "\cong"] \arrow[d, leftrightarrow, "\cong", "\text{wild nonabelian Hodge}"'] \arrow[dr, "\cong"] & \mathcal{M}_B \\
& \mathcal{M}_{Dol} \arrow[r, "\cong"] & \mathfrak{M}
\end{tikzcd}$$

$\mathcal{M}_B$: wild character variety (same dimension)

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"Nonabelian Hodge space"

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wild harmonic bundles (2d self-duality)

"Nonabelian Hodge space"

Classify via diagrams? (e.g. sometimes $\mathcal{M}^*$ is a quiver variety)

Much of the story can be summarised in the (slightly oversimplified) diagram:

$$
\begin{array}{ccccc}
\underline{\text{Dolbeault}} & & \underline{\text{De Rham}} & & \underline{\text{Betti}} \\[2ex]
\begin{array}{c}\text{Rational Lax matrices } L \\ \dot{L} = [P, L]\end{array} & & \begin{array}{c}\text{Rational diff. op.s} \\ \frac{d}{dz} - B\end{array} & & \begin{array}{c}\text{Stokes and} \\ \text{monodromy data}\end{array} \\[2ex]
\bigcap & & \bigcap & & \bigcap \\[2ex]
\begin{array}{c}\text{Mero. Higgs bundles} \\ (E, \Phi)\end{array} & \overset{\text{wnAbH}}{\longleftrightarrow} & \begin{array}{c}\text{Mero. connections} \\ (E, \nabla)\end{array} & \overset{\text{RHB}}{\longleftrightarrow} & \begin{array}{c}\text{Stokes local} \\ \text{systems}\end{array} \\[2ex]
\bigcup & & \bigcup & & \bigcup \\[2ex]
\begin{array}{c}\text{Holom. Higgs bundles} \\ (E, \Phi)\end{array} & \overset{\text{nAbH}}{\longleftrightarrow} & \begin{array}{c}\text{Holom. connections} \\ (E, \nabla)\end{array} & \overset{\text{RH}}{\longleftrightarrow} & \begin{array}{c}\text{Local systems/} \\ \pi_1\text{-rep.s}\end{array}
\end{array}
$$

$$
\begin{tikzcd}
 & \mathfrak{M} \arrow[dl, bend right, "1"', "\cong"] \arrow[d, "2"', "\cong"] \arrow[dr, bend left, dashed, "\cong"'] & \\
\mathcal{M}_{\mathrm{Dol}} \arrow[r, dotted, no head, "\mathrm{wnAbH}", "\cong"'] & \mathcal{M}_{\mathrm{DR}} \arrow[r, "\mathrm{RHB}", "\cong"'] & \mathcal{M}_{\mathrm{B}}
\end{tikzcd}
$$

FIGURE 2. Nonabelian Hodge space $\mathfrak{M}$, with three preferred algebraic structures.

En définitive, le problème posé au n° 4 exige l'intégration du système aux dérivées partielles (19), (20), ou, si l'on veut, avec les notations ordinaires:

$$(A_t)\qquad \begin{cases} \dfrac{\partial A^i_{hk}}{\partial t_j} = \displaystyle\sum_{l=1}^{m} \dfrac{A^i_{hl}A^j_{lk} - A^j_{hl}A^i_{lk}}{t_j - t_i} & (j \neq i), \\ & (h,\ k = 1, \ldots, m), \\ \displaystyle\sum_{j=1}^{n+2} \dfrac{\partial A^j_{hk}}{\partial t_i} = 0. & \end{cases}$$

C'est le système découvert par M. SCHLESINGER; il est complètement intégrable [car il a été déduit, moyennant une transformation (18) du système complètement intégrable (11)]; et sa solution dépend de $m^2(n+2)$ constantes arbitraires.

10. Je vais former maintenant le système dont l'intégration fait l'objet de ce Mémoire. Dans (A_t) remplaçons t_i par $\alpha_i + \varepsilon t_i$ $(i = 1, 2, \ldots, n)$; convenons toujours de prendre $\alpha_{n+1} = 0$, $\alpha_{n+2} = 1$, et remplaçons A^i par $\varepsilon^{-1}A^i$ $(i = 1, \ldots, n+2)$. Puis faisons tendre ε vers 0. A la limite, le système (A_t) deviendra

$$(A_\alpha)\qquad \begin{cases} \dfrac{\partial A^i_{hk}}{\partial t_j} = \displaystyle\sum_{l=1}^{m} \dfrac{A^i_{hl}A^j_{lk} - A^j_{hl}A^i_{lk}}{\alpha_j - \alpha_i} & (j \neq i), \\ \displaystyle\sum_{j=1}^{n+2} \dfrac{\partial A^j_{hk}}{\partial t_i} = 0 & (h,\ k = 1, \ldots, m). \end{cases}$$

D'après la façon même dont on l'a déduit de (A_t), le système (A_α) est le *simplifié* de

p.163 of: R. Garnier, Sur une classe de systèmes différentiels Abéliens déduits de la théorie des équations linéaires, Rend. Circ. Mat. Palermo 43 (1919), 155–191.

$$a_{hk} = \sum_{i=1}^{n+2} \frac{A_{hk}^i}{x - \alpha_i};$$

et nous poserons

$$a_{hk} \equiv \frac{b_{hk}(x)}{\varphi(x)}$$

de sorte que $b_{hk}(x)$ sera un polynome en x, de degré $n+1$ au plus, le coefficient de x^{n+1} étant indépendant des t_i, en vertu des équations $(A_\alpha)_2$.

Enfin, de toute l'analyse développée au Chapitre I nous ne retiendrons, pour l'intégration de (A_α), qu'une seule formule, à la vérité d'importance capitale: à savoir la *simplifiée* de (11) pour $j = 0$; cette simplifiée s'écrit avec les notations actuelles:

$$\frac{\partial b}{\partial t_i} \div \frac{A^i b - b A^i}{x - \alpha_i} \qquad (i = 1, \ldots, n). \tag{21}$$

Cela étant, je vais établir deux propositions préliminaires sur lesquelles repose toute l'intégration du système (A_α).

12. Théorème fondamental (I). — *Si dans la relation* [12])

$$f(x, y) \equiv \begin{vmatrix} b_{11}(x) + y & b_{12}(x) & \ldots & b_{1m}(x) \\ b_{21}(x) & b_{22}(x) + y & \ldots & b_{2m}(x) \\ \ldots & \ldots & \ldots & \ldots \\ b_{m1}(x) & b_{m2}(x) & \ldots & b_{mm}(x) + y \end{vmatrix} = 0 \tag{22}$$

on remplace les A_{hk}^i qui figurent dans b_{hk} par des intégrales de (A_α), la courbe algébrique $f(x, y) = 0$ ainsi obtenue a tous ses coefficients indépendants des t_i.

p.164 of: R. Garnier, Sur une classe de systèmes différentiels Abéliens déduits de la théorie des équations linéaires, Rend. Circ. Mat. Palermo 43 (1919), 155–191.

THEOREM 1. *There is a one-to-one correspondence between*

(i) *a polynomial* $A = \sum_{s=0}^{\nu} A_s h^s$ *with matrix coefficients (modulo conjugation by complex diagonal matrices), having the propertie.* $A_\nu = \mathrm{diag}(a_1, \ldots, a_n)$, $a_i \in \mathbb{C}^*$, $\prod_{i<j}(a_i - a_j) \neq 0$, *and* $(A_{\nu-1})_{1,k} \neq 0$ $(k \neq 1)$; *moreover* A *has in the limit* $h \to 0$ *distinct eigenvectors all not perpendicular*[11] *to* e^k *for some* k.

(ii) *a curve* X *of genus* $g = (n(n-1)/2)\nu - n + 1$ *with* $2n$ *distinct points* $P_1, \ldots, P_n$, $Q_1, \ldots, Q_n$ *and a general positive divisor* $\mathscr{D}$ *on* X *of degree* g *not containing any of the points* P_i *or* Q_i; *the points above have the following properties: for some meromorphic functions* h *and* z *on* X

$$(h) = -\sum_1^n P_i + \sum_1^n Q_i$$

and

$$(z) = -\nu \sum_1^n P_i + n\nu \text{ zeros, distinct from the } P_i .$$

Moreover any polynomial function $u = P(z, h, h^{-1})$ *on* X *leads to an isospectral deformation of* A

$$\dot{A} = [A, P(A, h, h^{-1})_+],$$

where $P(A, h, h^{-1})_+$ *denotes the polynomial part (in* h*) of* $P(A, h, h^{-1})$. *The flow above is a linear flow on* $\mathrm{Jac}(X)$ *defined by*

$$\sum_{i=1}^{g} \int_{\nu_i}^{\nu_i(t)} \omega = \sum_{l=1}^{n} \mathrm{Res}_{P_l}(\omega u) t.$$

In particular the flows (cf. I (4.43))

$$\dot{A} = [A, (f'(Ah^{-j})h^{k-j})_+]$$

are linear flows on $\mathrm{Jac}(X)$; *they are equivalent to one of the polynomial flows above.*

Proof. As a first step, the curve X, defined by the algebraic equation

$$Q(z, h) \equiv \det(A - zI),$$

is shown to have the properties stated in (ii). For z, h large,

$$Q(z, h) = \prod_1^n (a_i h^\nu - z) + C_1 h^{n\nu-1} + C_2 z^{n-1} + \text{lower-order terms},$$

[11] Dubrovin *et al.* [6] have considered similar matrix polynomials.

Adler–Van Moerbeke 1980

Commun. math. Phys. 76, 65–116 (1980)

Communications in Mathematical Physics

Monodromy- and Spectrum-Preserving Deformations I

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Abstract. A method for solving certain nonlinear ordinary and partial differential equations is developed. The central idea is to study monodromy preserving deformations of linear ordinary differential equations with regular and irregular singular points. The connections with isospectral deformations and with classical and recent work on monodromy preserving deformations are discussed. Specific new results include the reduction of the general initial

Simplified Schlesinger's Systems.

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(ricevuto il 10 Settembre 1979)

It is known that classical Painlevé transcendents I-II in the limit $t \to a + \varepsilon t_1 : \varepsilon \to 0$ turn to be Weierstrass or Jacobi elliptic functions (1). GARNIER (2) made a careful analysis of more general transcendents like Painlevé I-VI and their generalizations. It is natural to presume that the most general isomonodromy deformation equations —Schlesinger's equations (3)—in the limit $t_i \to a_i + \varepsilon t_{1i} : \varepsilon \to 0$ are reduced to some classical completely integrable systems.

Very good connections

~ models in Biquard–B. 2004

(cf. exposition in {arXiv:1203.6607, arXiv:1703.10376})

Σ compact Riemann surface, $\underset{\sim}{a} \subset \Sigma$ finite subset

$V \to \Sigma$ holomorphic vector bundle

$\exists$ parabolic filtrations (in V_a $\forall$ $a \in \underset{\sim}{a}$)

$\nabla : V \to V \otimes \Omega^1(* \underset{\sim}{a})$ meromorphic connection

such that ...

Very good connections

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Σ compact Riemann surface, $\underset{\sim}{a} \subset \Sigma$ finite subset

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$\mathcal{F}$ parabolic filtrations (in V_a $\forall a \in \underset{\sim}{a}$)

$\nabla : V \to V \otimes \Omega^1(*\underset{\sim}{a})$ meromorphic connection

such that have local bases (at each $a \in \underset{\sim}{a}$) splitting $\mathcal{F}_a$ such that:

- $\nabla = d - A$, $A = dQ + \Lambda \frac{dz}{z}$ + holomorphic terms

  $Q = \sum_1^k \frac{A_i}{z^i}$, A_i diagonal matrices (irregular type)

  $\Lambda \in \mathfrak{h}$ preserves $\mathcal{F}_a$, $\mathfrak{h} = \mathrm{Lie}(H)$, $H = C_G(Q)$

["Good" if some local cyclic pullback is very good (twisted case)]

$\rightsquigarrow \mathcal{M}_{DR}$ moduli of stable connections, $\underset{\sim}{Q}$, $Gr(\Lambda)$, parabolic weights fixed

Very good Higgs bundles

~ models in Biquard-B. 2004

(cf. exposition in {arXiv:1203.6607, arXiv:1703.10376})

Σ compact Riemann surface, $\underset{\sim}{a} \subset \Sigma$ finite subset

$V \to \Sigma$ holomorphic vector bundle

$\mathcal{F}$ parabolic filtrations (in V_a $\forall a \in \underset{\sim}{a}$)

$\Phi : V \to V \otimes \Omega^1(* \underset{\sim}{a})$ meromorphic Higgs field

such that have local bases (at each $a \in \underset{\sim}{a}$) splitting $\mathcal{F}_a$ such that:

- $$\Phi = dQ + \Lambda \frac{dz}{z} + \text{holomorphic terms}$$

$$Q = \sum_1^k \frac{A_i}{z^i}, \quad A_i \text{ diagonal matrices} \quad \text{(irregular type)}$$

$$\Lambda \in \mathfrak{h} \text{ preserves } \mathcal{F}_a, \quad \mathfrak{h} = \mathrm{Lie}(H),\ H = C_G(Q)$$

["Good" if some local cyclic pullback is very good (twisted case)]

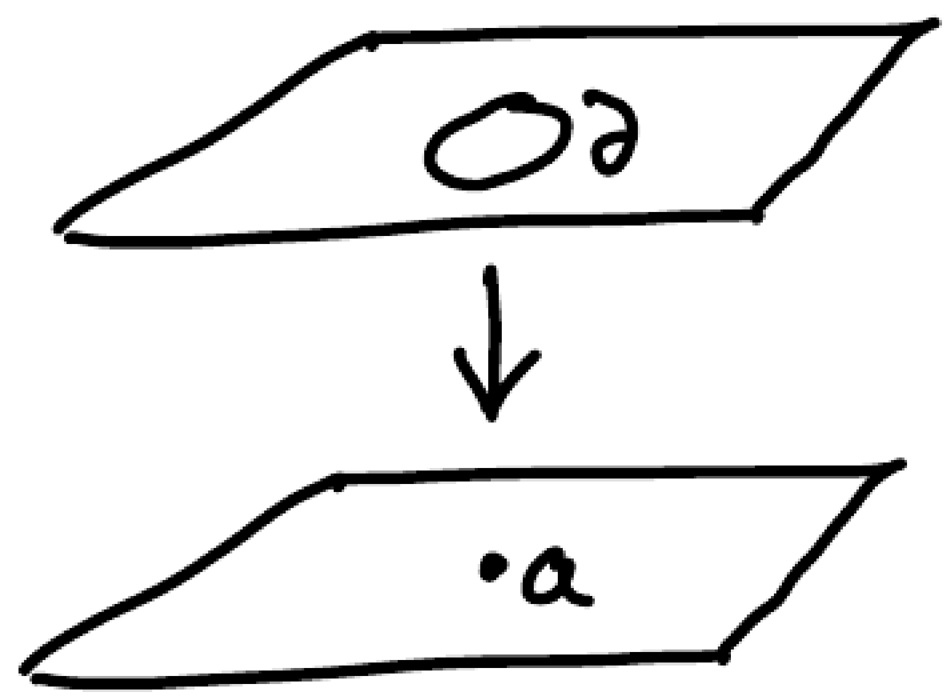
$\leadsto \mathcal{M}_{DoL}$ moduli of stable Higgs bundles, $\underset{\sim}{Q}$, $Gr(\Lambda)$, parabolic weights fixed

General choices / boundary data (twisted case) [Betti weights zero]

<u>Fact</u> $\exists$ covering $\mathcal{I} \to \partial$ such that:

$$\left\{ \begin{matrix} \text{connections on formal} \\ \text{punctured disk} \end{matrix} \right\} \Leftrightarrow \left\{ \begin{matrix} \mathcal{I}\text{-graded local systems} \\ \text{of vector spaces} \end{matrix} \right\}$$

[Fabry, Hukuhara, Turrittin, Levelt, Jurkat, Deligne]



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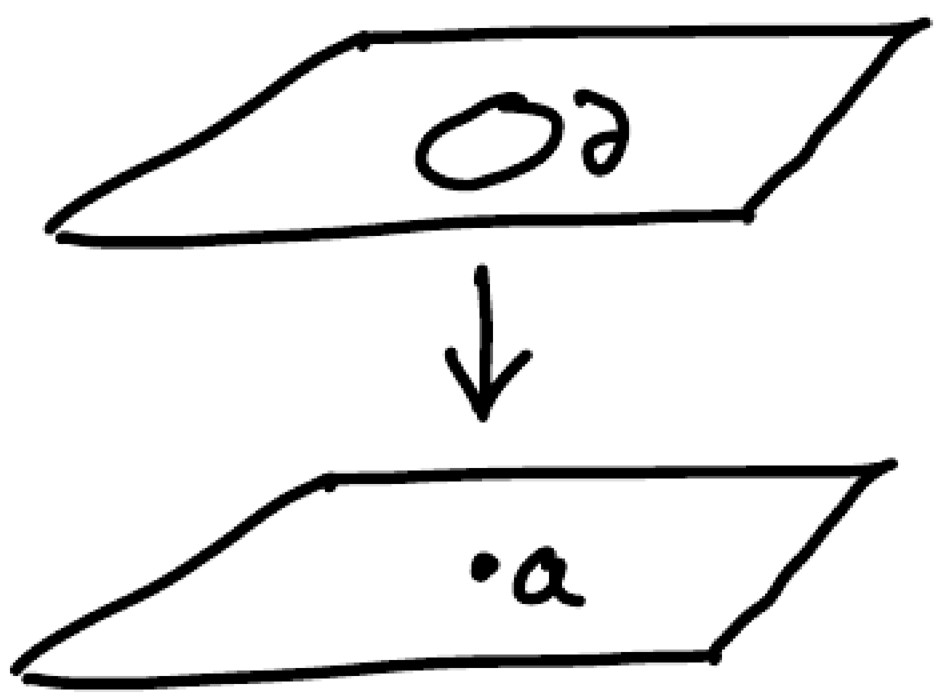

function on sector: $q = \sum_{i \geq 0} a_i z^{-i/r}$ $(r \in \mathbb{N})$

$\Rightarrow$ Stokes circle $\langle q \rangle \to \partial$ (Riemann surface / Galois orbit)

$$\boxed{\begin{array}{l} \mathcal{I} = \bigcup \langle q \rangle \\ \downarrow \qquad \text{exponential local system} \\ \partial \end{array}}$$

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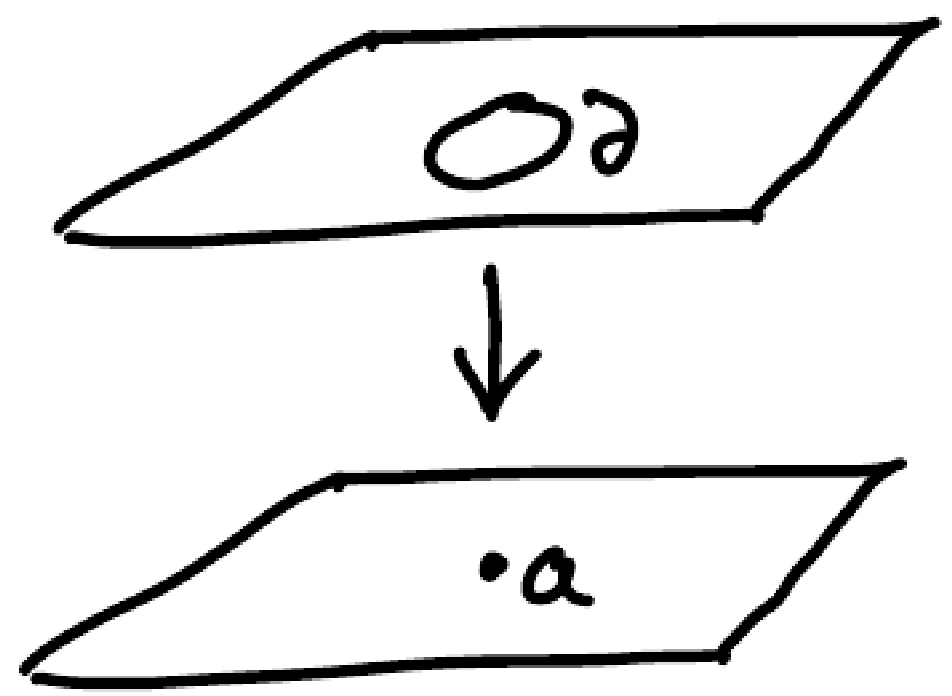

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$\mathcal{I}$-graded local system $V \to \partial$ of vector spaces

$\Leftrightarrow$ local system $V \to \mathcal{I}$ with compact support

i.e. $V \to I$, $I \subset \mathcal{I}$ finite subcover

$\Rightarrow$ Irregular class $\Theta = n_1 \langle q_1 \rangle + \dots + n_m \langle q_m \rangle$ $\quad n_i = \mathrm{rk}\, V|_{\langle q_i \rangle}$

$+$ monodromy classes $\quad \mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$

In simple examples this growth/decay can be easily visualised in the Stokes diagram, as in the example of $q = x^{17}$ in Figure 5, where the singularity is at $a = \infty$ (so $z = x^{-1}$ is a local coordinate vanishing at a). For example we see on the positive real axis that the function $\exp(x^{17})$ has maximal growth there, and there are 16 other evenly spaced directions of maximal growth, interlaced with 17 directions of maximal decay, the first at $\arg(x) = \pi/17$.

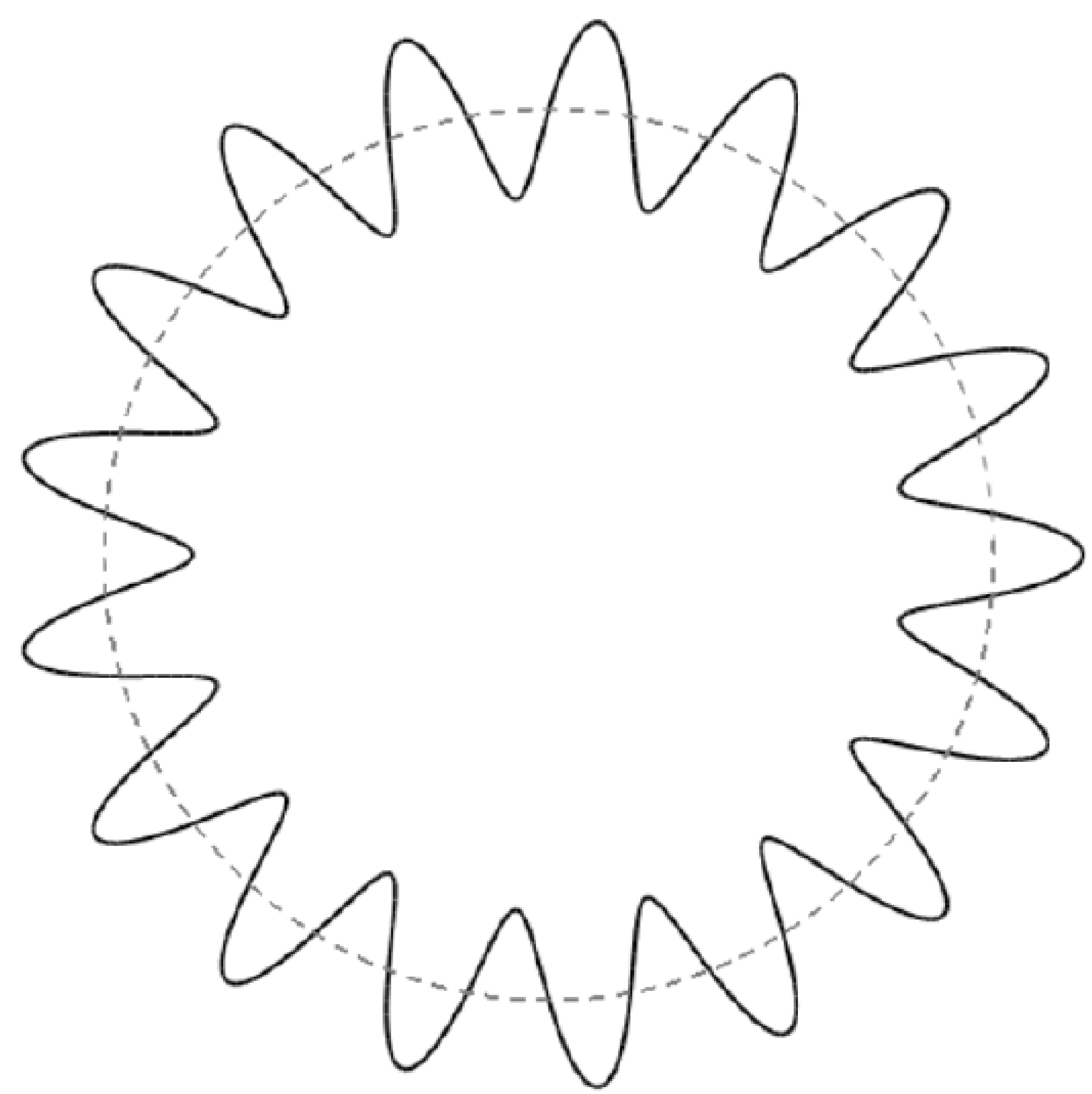


FIGURE 5. Stokes diagram for $\langle x^{17}\rangle$: the Stokes circle $\langle x^{17}\rangle$ is projected to the plane so as to indicate the growth/decay of $\exp(x^{17})$ near ∞.

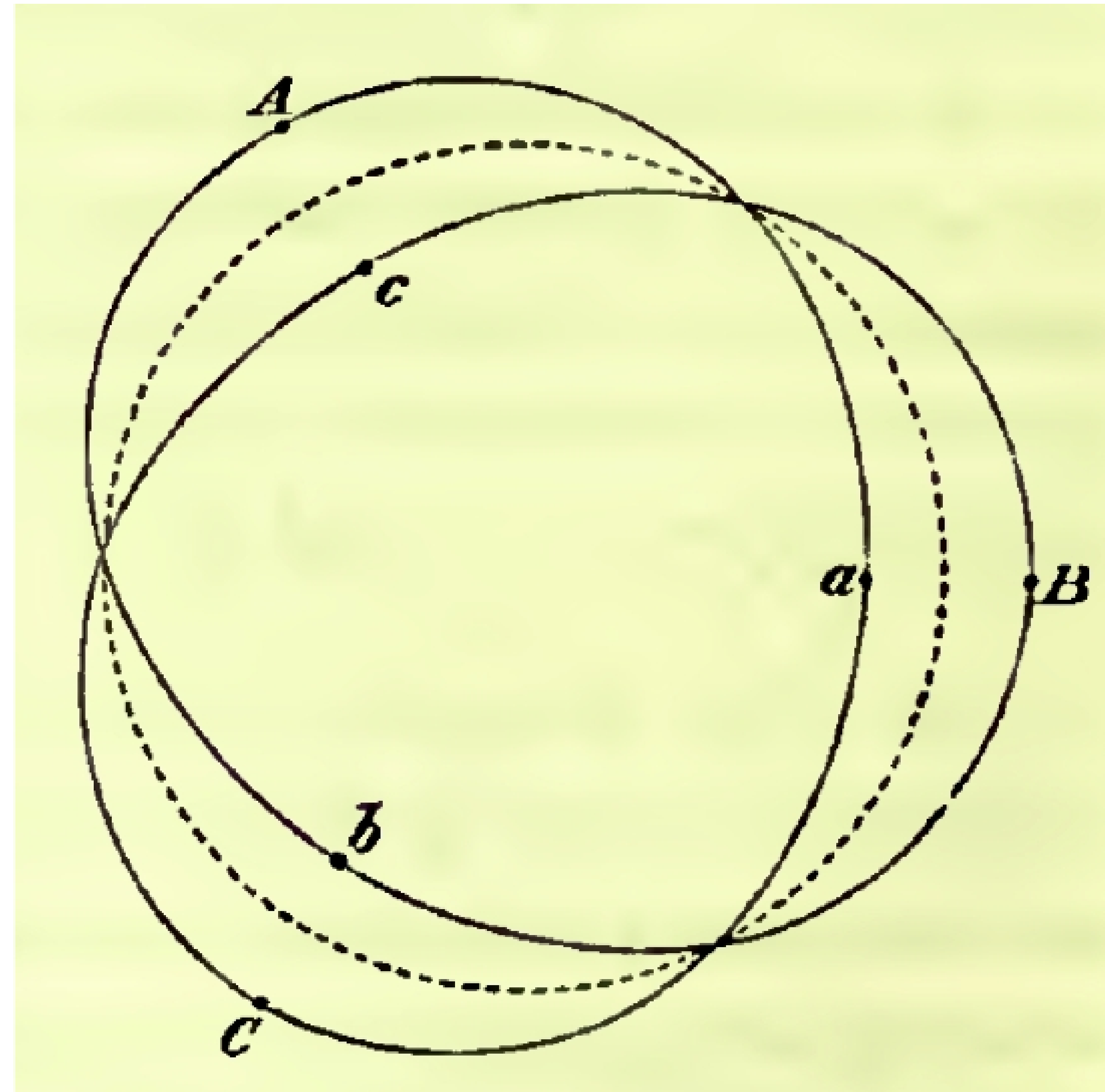
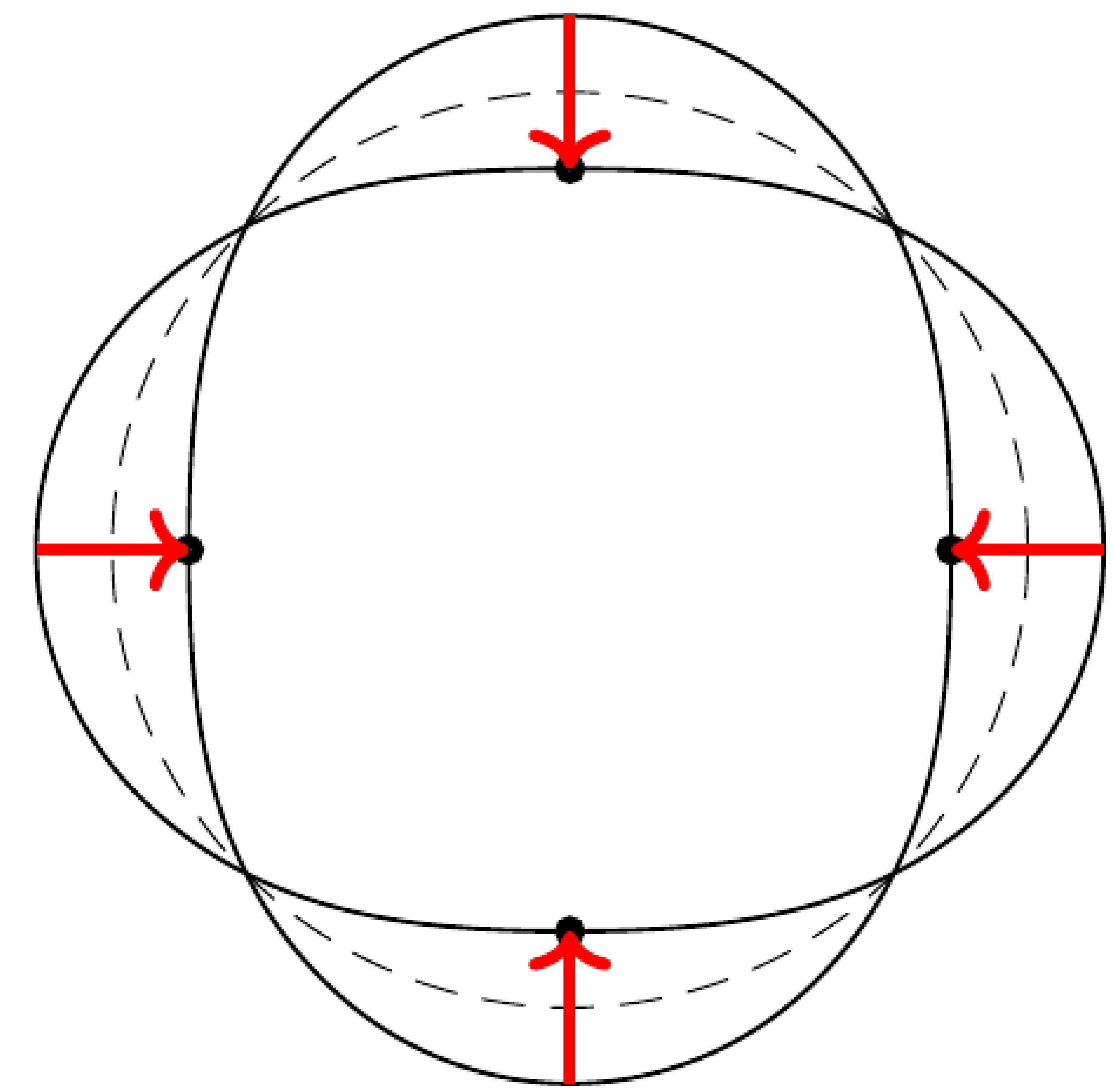


FIGURE 6. The Stokes diagram of $\langle 2x^{3/2}\rangle$, from Stokes' paper [?] on the Airy equation. The points a, b, c are the points of maximal decay.



Stokes diagram of the Weber equation, with Stokes arrows drawn.

There is a javascript program here:

https://webusers.imj-prg.fr/~philip.boalch/stokesdiagrams.html

to draw lots of other examples of Stokes diagrams, the Stokes diagrams of the "symmetric" or "hypotrochoid" irregular classes $I(a:b)$ (see the explanation in the box at the bottom there).[15] In brief $I(a:b)$ is the pull-back to the x-plane of the irregular class $\langle w^{1/b}\rangle$ under the map $w = x^a$. It has k Stokes circles where $k = (a, b)$ is the highest common factor. Explicitly:

$$I(a:b) = \bigsqcup_{i=0}^{k-1} \langle \varepsilon^i x^{a/b} \rangle \subset \mathcal{I}$$

where $\varepsilon = \exp(2\pi i/b)$. For example it is the irregular class at $x = \infty$ of the Molins–Turrittin equation $y^{(b)} = x^\nu y$, if $a = \nu + b$ [?, ?]. Upto a constant $I(1:q+1)$ is also the irregular class at ∞ of the differential equation for the hypergeometric series ${}_0F_q$.

10.5. **Rank two examples.** The simplest rank two Stokes diagrams are collected in Figure 7. The left four are *rigid* in that their (symplectic) wild character varieties are dimension zero. They come from the ODEs of Clifford, Airy, Whittaker, Hermite–Weber. The next two, with 5 or 6 crossings, give the wild character varieties of Painlevé I and II.

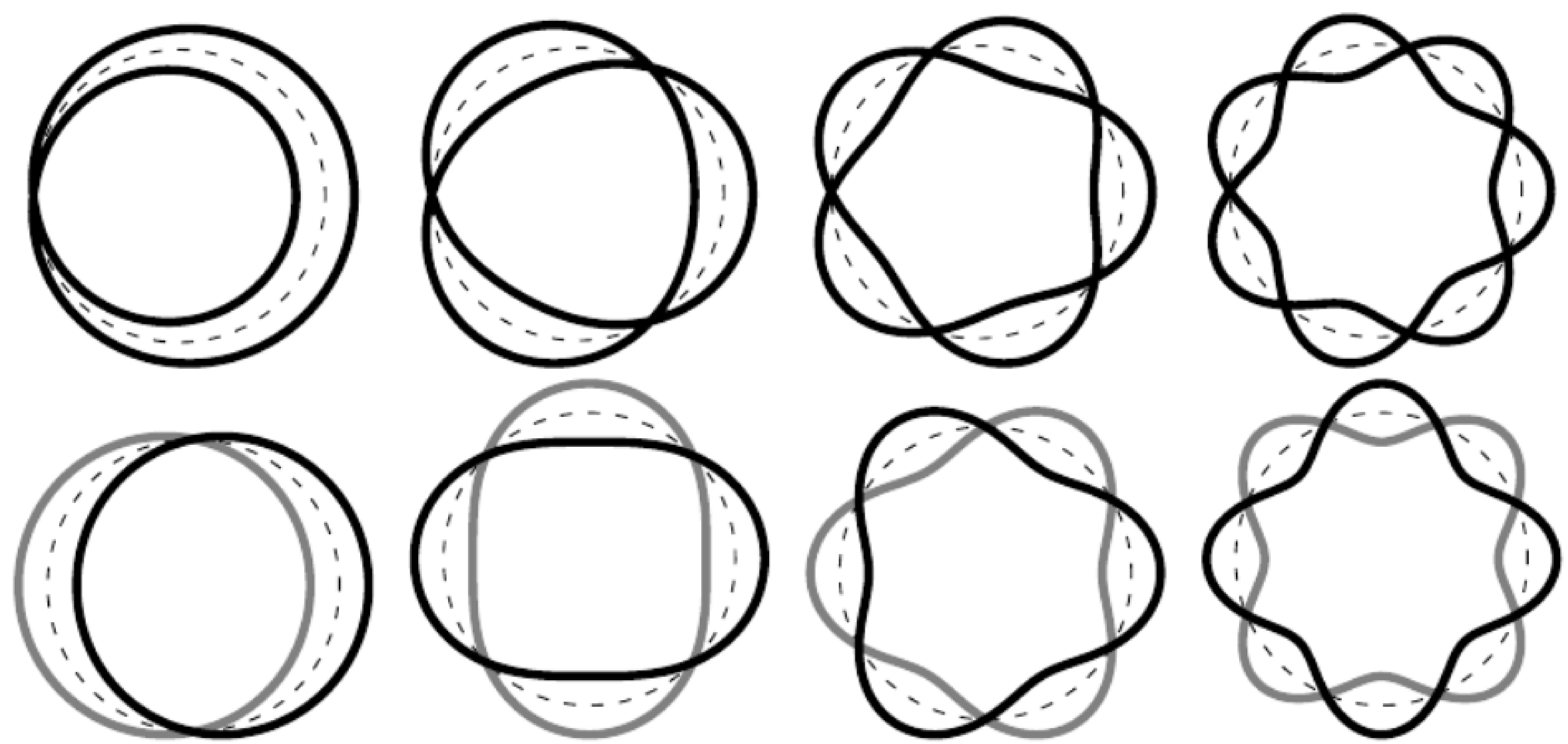


FIGURE 7. The simplest rank two Stokes diagrams $I(k:2), k = 1, 2, \ldots, 8$.

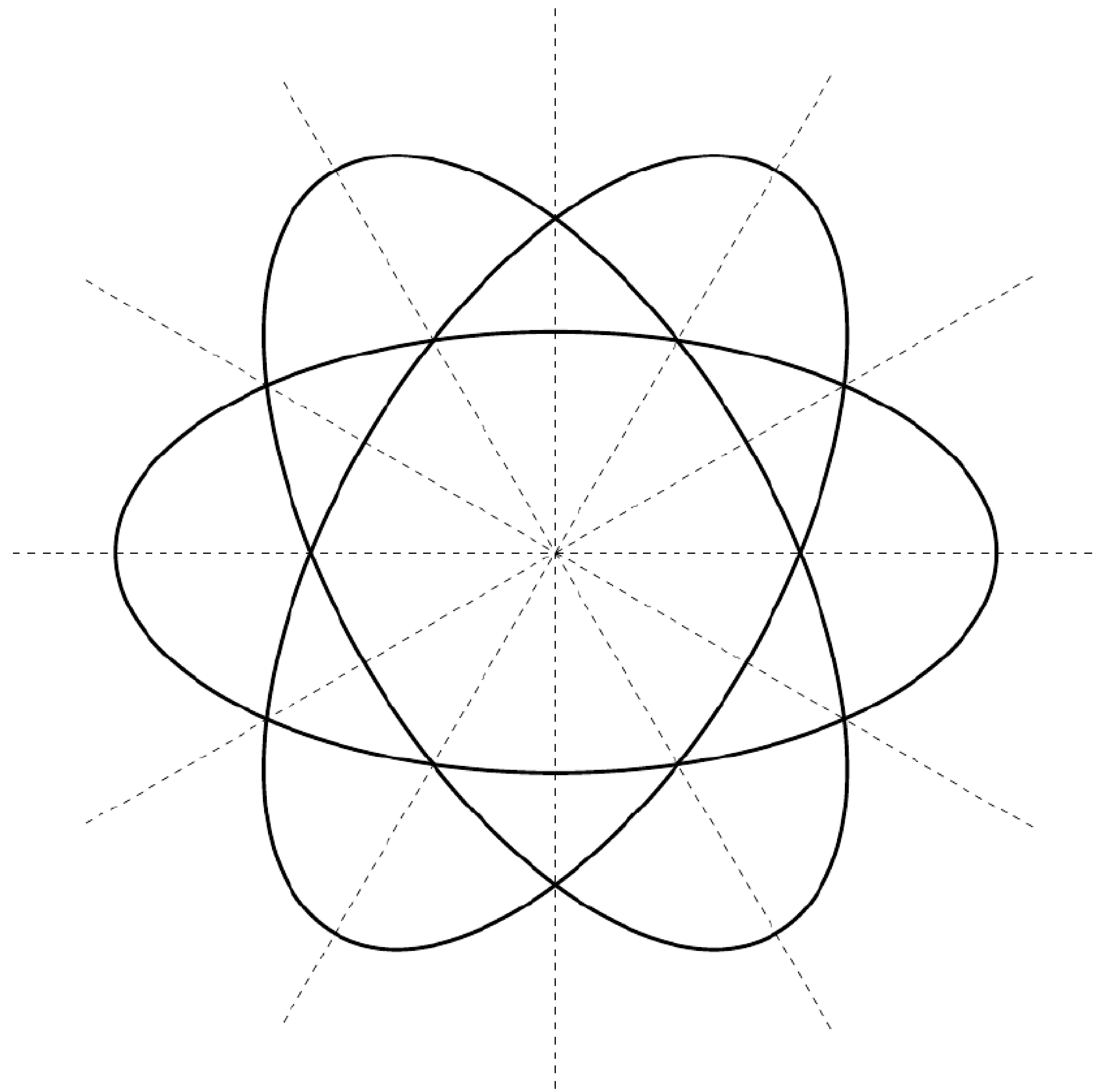


FIGURE 8. Example rank three Stokes diagram, $I(6:3)$.

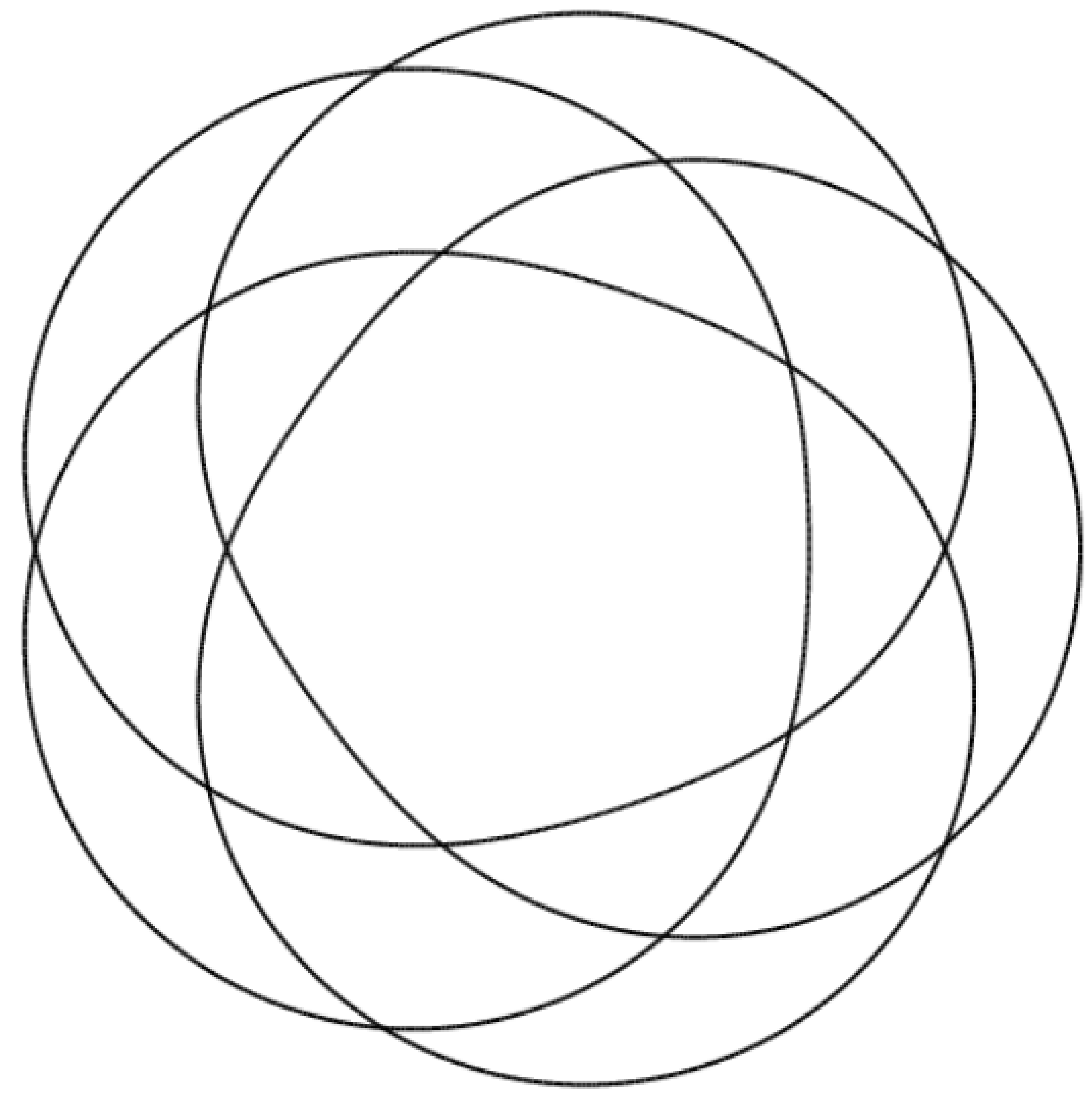


FIGURE 9. Stokes diagram at ∞ for the "hyperairy" equation $y^{(4)} = xy$

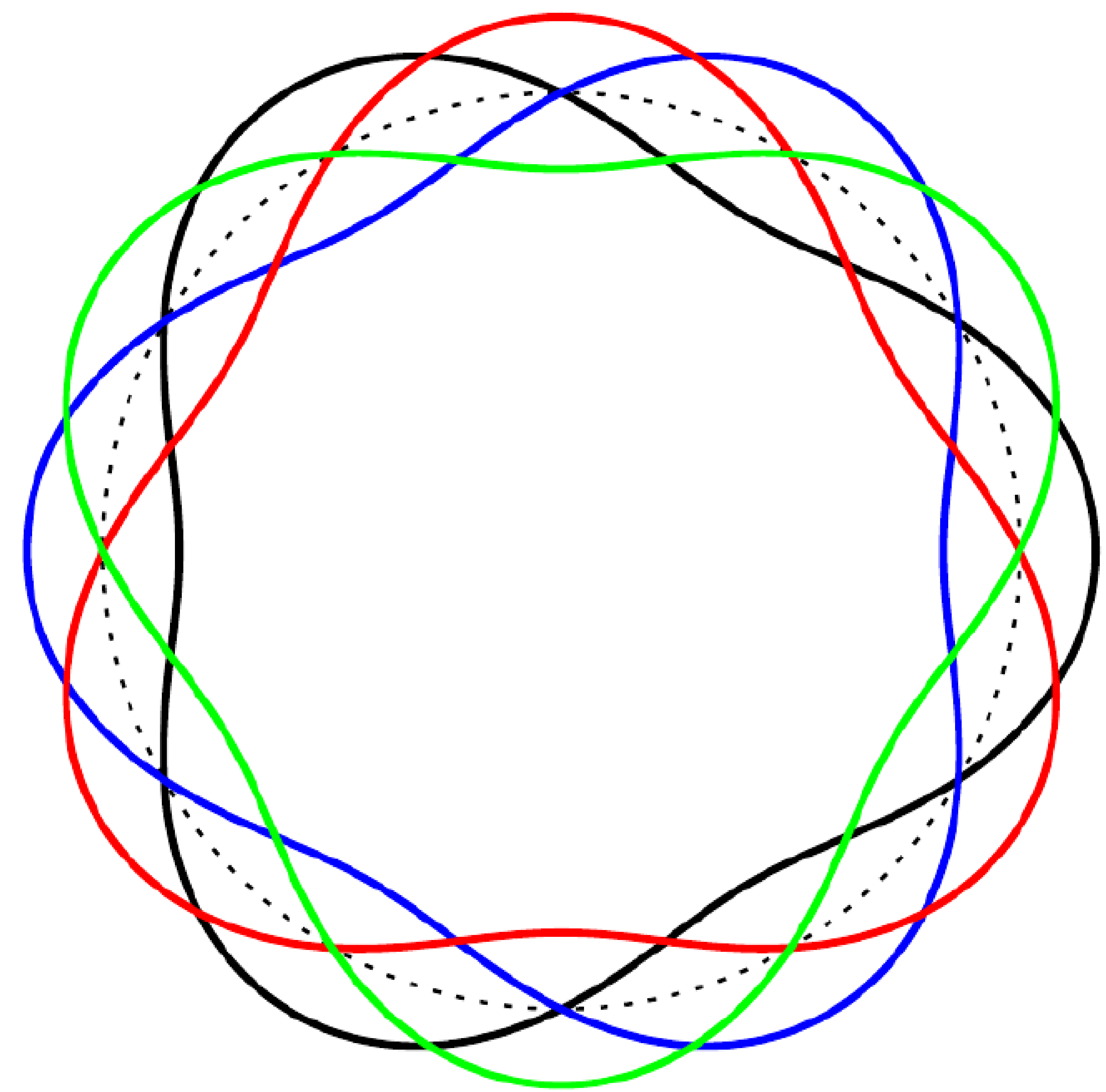


FIGURE 10. Another example rank four Stokes diagram, $I(12\!:\!4)$.

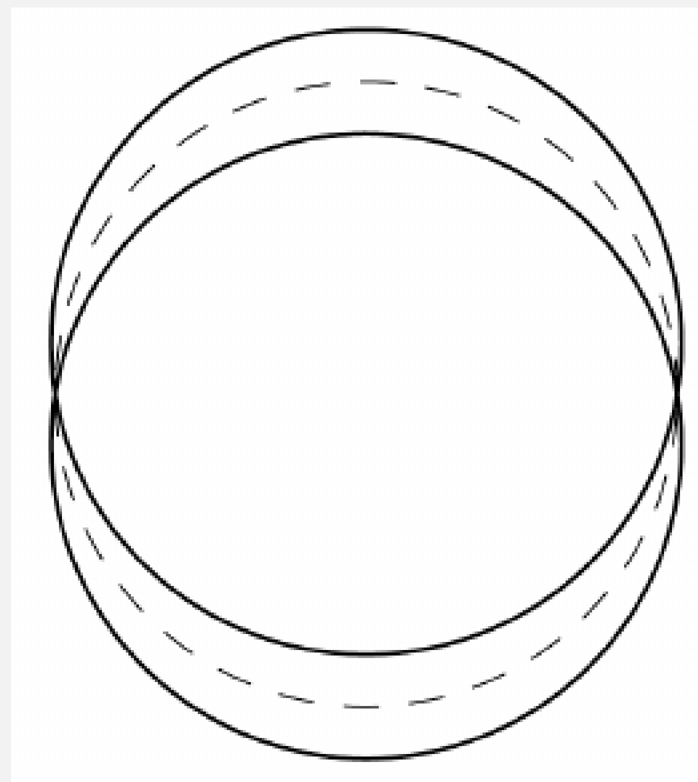
10.6. **Example Stokes diagrams: Bessel's equation.**
Bessel's differential equation is

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

where $\alpha \in \mathbb{C}$. This has a regular singularity at 0 and an irregular singularity at ∞. A short computation, or a glance at a book, shows that the irregular class $x = \infty$ is:

$$\Theta = \langle ix \rangle + \langle -ix \rangle$$

and that α determines the local monodromy eigenvalues at 0. In particular the singular directions are the two halves of the imaginary axis.



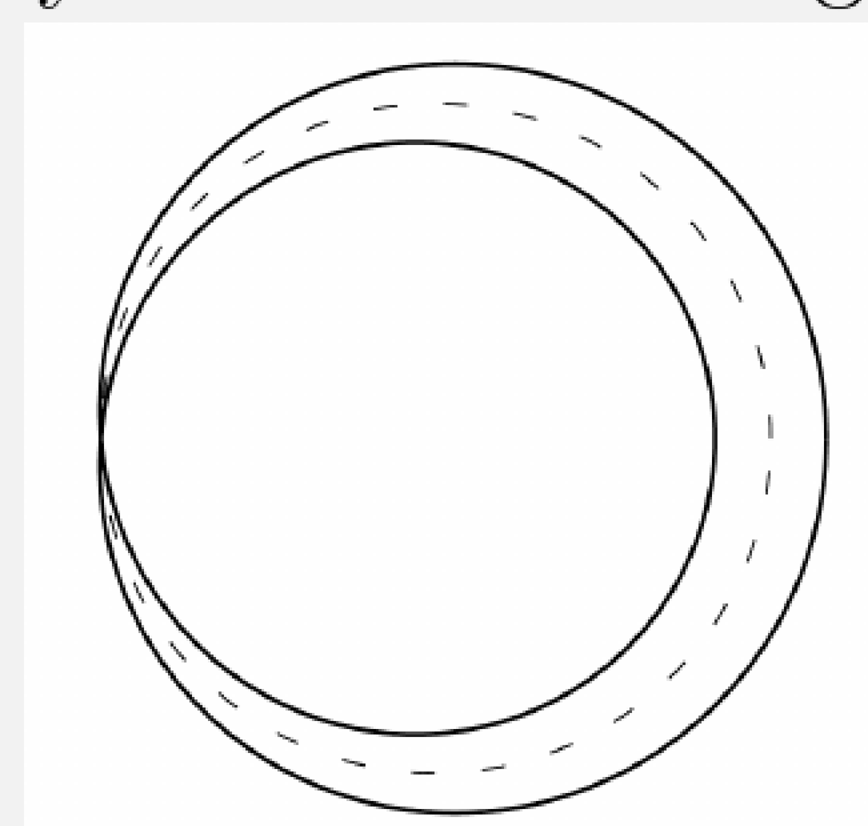
10.7. **Example Stokes diagrams: Bessel–Clifford equation.** The Bessel–Clifford equation (also known as the confluent hypergeometric limit equation, Kummer's second equation, or the ${}_0F_1$ equation) is:

$$xy'' + ay' = y. \tag{10.1}$$

If f is any solution of this, then $x^{a-1} \cdot f(-x^2/4)$ solves the Bessel equation with parameter $\alpha = a - 1$. The irregular class at $x = \infty$ is

$$\langle 2x^{1/2} \rangle$$

and (if $a \notin \mathbb{Z}$) the monodromy around 0 has eigenvalues $1, \exp(-2\pi i a)$.



9.5. **Wild Riemann surfaces.** The irregular class makes up the basic "new modular parameters" that occur for irregular connections, behaving just like the modulus of the underlying Riemann surface and the location of the marked points $\mathbf{a}$.

In particular it behaves completely differently to the formal residue Λ.

This motivates the following definition:

Definition 9.5. *A rank n wild Riemann surface is a triple $\mathbf{\Sigma} = (\Sigma, \mathbf{a}, \Theta)$ where Σ is a Riemann surface, $\mathbf{a} \subset \Sigma$ is a finite subset and $\Theta = \{\Theta_a \mid a \in \mathbf{a}\}$ is the data of a rank n irregular class at each point $a \in \mathbf{a}$.*

Here we are mainly interested in the case where Σ is compact. We will define the character variety $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$ of any such wild Riemann surface, show that it is Poisson and forms a local system of varieties under any admissible deformation of $\mathbf{\Sigma}$.

Of course if all the irregular classes are trivial then $\mathbf{\Sigma} = (\Sigma, \mathbf{a}, \Theta)$ just amounts to choosing a Riemann surface with some marked points, and then $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$ will be the usual (tame) character variety defined previously $\cong \mathrm{Hom}(\pi_1(\Sigma^\circ, b), \mathrm{GL}_n(\mathbb{C}))/\mathrm{GL}_n(\mathbb{C})$.

Notes: This definition is from [B2014] Defn 8.1, Rmk 10.6, [BY2015] §4. There are several minor variations that we won't worry about here, but are sometimes useful: One can work with irregular types instead of irregular classes (which were called "bare irregular types" in [B2014] Rmk 10.6); this is analogous to whether or not we order the points $\mathbf{a}$. Also one can work with smooth complex algebraic curves instead of Riemann surfaces (which doesn't make much difference in the compact case); the terms "irregular curve" or "wild curve" are sometimes used to replace the term "wild Riemann surface" in the algebraic case. Op. cit. give the definition for any complex reductive group, not just $\mathrm{GL}_n(\mathbb{C})$.

Séminaire BOURBAKI
(Mai 1958)

MODULES DES SURFACES DE RIEMANN

par André WEIL

Par la combinaison des idées (récentes) de KODAIRA et SPENCER sur la variation des structures complexes avec les idées (anciennes) de TEICHMÜLLER sur le problème des modules, la théorie a fait dernièrement quelques progrès qu'on se propose d'exposer ici.

Soit T_o une surface orientée compacte de genre g, donnée une fois pour toutes. Par une surface de Riemann de genre g, on entend, comme d'habitude, une variété complexe compacte de dimension complexe 1, de genre g, munie de son orientation naturelle. Par une surface de Teichmüller de genre g, on entendra une surface de Riemann S de genre g, munie de plus d'une classe (au sens de l'homotopie) d'applications de T_o dans S, classe dont on suppose qu'elle contient au moins un homéomorphisme conservant l'orientation ; c'est là une structure (plus "riche" que celle de structure de surface de Riemann). Si π^o désigne le

Il est utile de définir une notion intermédiaire entre celle de surface de Riemann et celle de surface de Teichmüller : on l'obtient en se donnant les images des A_i^o, non dans $\pi(S)$, mais dans $H_1(S)$; la donnée de ces images sur la surface de Riemann S détermine ce qu'on appellera une "surface de Torelli". Au

Nonabelian Hodge theory on wild Riemann surfaces

Let $\mathbf{\Sigma} = (\Sigma, \mathbf{a}, \Theta)$ be a rank n wild Riemann surface whose underlying Riemann surface Σ is compact. Choose some residue data $\mathbf{R}$ for $\mathbf{\Sigma}$ of (global) degree zero. Recall that a "connection on $\mathbf{\Sigma}$" means a good meromorphic connection on a parabolic vector bundle on Σ with poles/parabolic filtrations at $\mathbf{a}$, and irregular class Θ_a at each point $a \in \mathbf{a}$. Similarly for Higgs bundles on $\mathbf{\Sigma}$.

Let $\mathcal{M}_{\mathrm{DR}}(\mathbf{\Sigma}, \mathbf{R})$ be the holomorphic moduli space of stable connections on $\mathbf{\Sigma}$ with residue data $\mathbf{R}$. Similarly let $\mathcal{M}_{\mathrm{Dol}}(\mathbf{\Sigma}, \mathbf{R})$ be the holomorphic moduli space of stable Higgs bundles on $\mathbf{\Sigma}$ with residue data $\mathbf{R}$. We suppose that the boundary data is chosen so they are not empty.

Theorem 1.1 (Biquard–B. 2004)**.** *There is a hyperkähler manifold $\mathfrak{M}(\mathbf{\Sigma}, \mathbf{R})$ (equipped with a family of complex structures parameterised by $\mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$) that is a moduli space of irreducible wild harmonic bundles on $\Sigma^\circ = \Sigma \setminus \mathbf{a}$ with boundary conditions determined by $\mathbf{\Sigma}, \mathbf{R}$ such that:*
1) In the complex structure determined by $1 \in \mathbb{P}^1$ the space $\mathfrak{M}(\mathbf{\Sigma}, \mathbf{R})$ is isomorphic as a complex manifold to the moduli space $\mathcal{M}_{\mathrm{DR}}(\mathbf{\Sigma}, \mathbf{R})$ of stable good meromorphic connections,
2) In the complex structure determined by $0 \in \mathbb{P}^1$ the space $\mathfrak{M}(\mathbf{\Sigma}, \mathbf{R})$ is isomorphic as a complex manifold to the moduli space $\mathcal{M}_{\mathrm{Dol}}(\mathbf{\Sigma}, \mathbf{R})$ of stable good meromorphic Higgs bundles,
3) If the residue data $\mathbf{R}$ is semisimple and there are no strictly semistable connections on $\mathbf{\Sigma}$ with residue data $\mathbf{R}$, then the hyperkähler metric on $\mathfrak{M}(\mathbf{\Sigma}, \mathbf{R})$ is complete.

The boundary data is related by the following table:

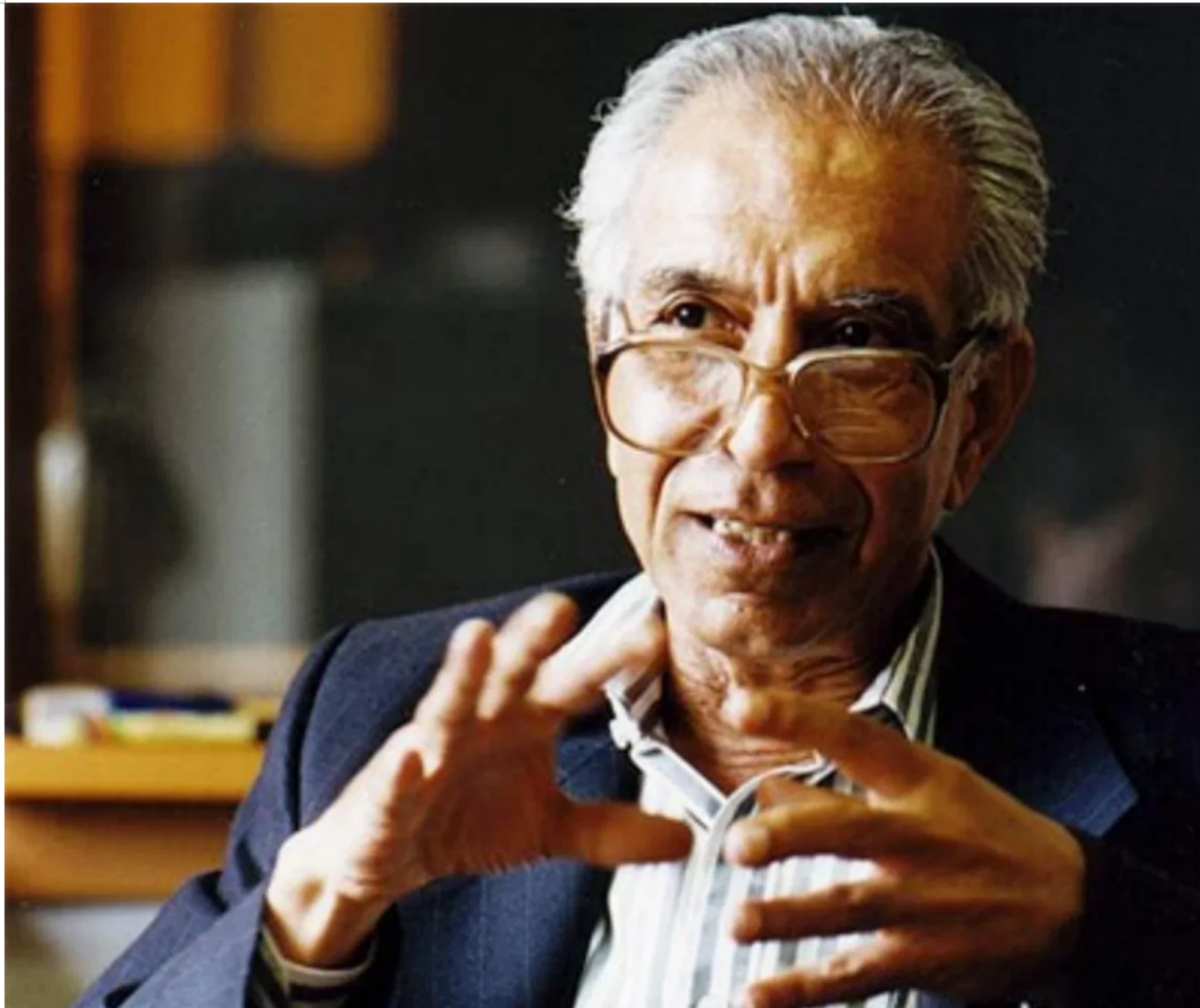
	Dolbeault	De Rham	Betti
weights $\in [0,1), [0,1), \mathbb{R}$	$\lceil\tau\rceil - \tau$	θ	$\phi = \theta + \tau$
eigenvalues $\in \mathbb{C}, \mathbb{C}, \mathbb{C}^*$	$\frac{1}{2}(\phi + \sigma)$	$\lambda = \tau + \sigma$	$\mu = \exp(2\pi i\lambda)$
exponential factors	$\frac{1}{2}q$	q	$\langle q\rangle$

- In tame case ($q = 0$) most of this is due to Konno 1993 and Nakajima 1996 (using Biquard's weighted Sobolev space approach), strengthening Simpson's 1990 tame bijective correspondence in to a diffeomorphism. Even then the completeness statement (beyond the finite energy "strongly parabolic" setting in Konno's paper) is new.

- In the wild case the construction of harmonic bundles from irreducible irregular connections on meromorphic bundles (i.e. Betti weights zero) was established earlier by Sabbah 1999.

- In the nonsingular/compact case ($q = 0 = \lambda = \theta$) it is due to Hitchin, Donaldson, Corlette, Simpson, (Fujiki, Diederich–Ohsawa).

- If also the Higgs field is zero this gives the Narasimhan–Seshadri theorem.



Actes, Congrès intern. Math., 1970. Tome 2, p. 199 à 201.

GEOMETRY OF MODULI SPACES OF VECTOR BUNDLES

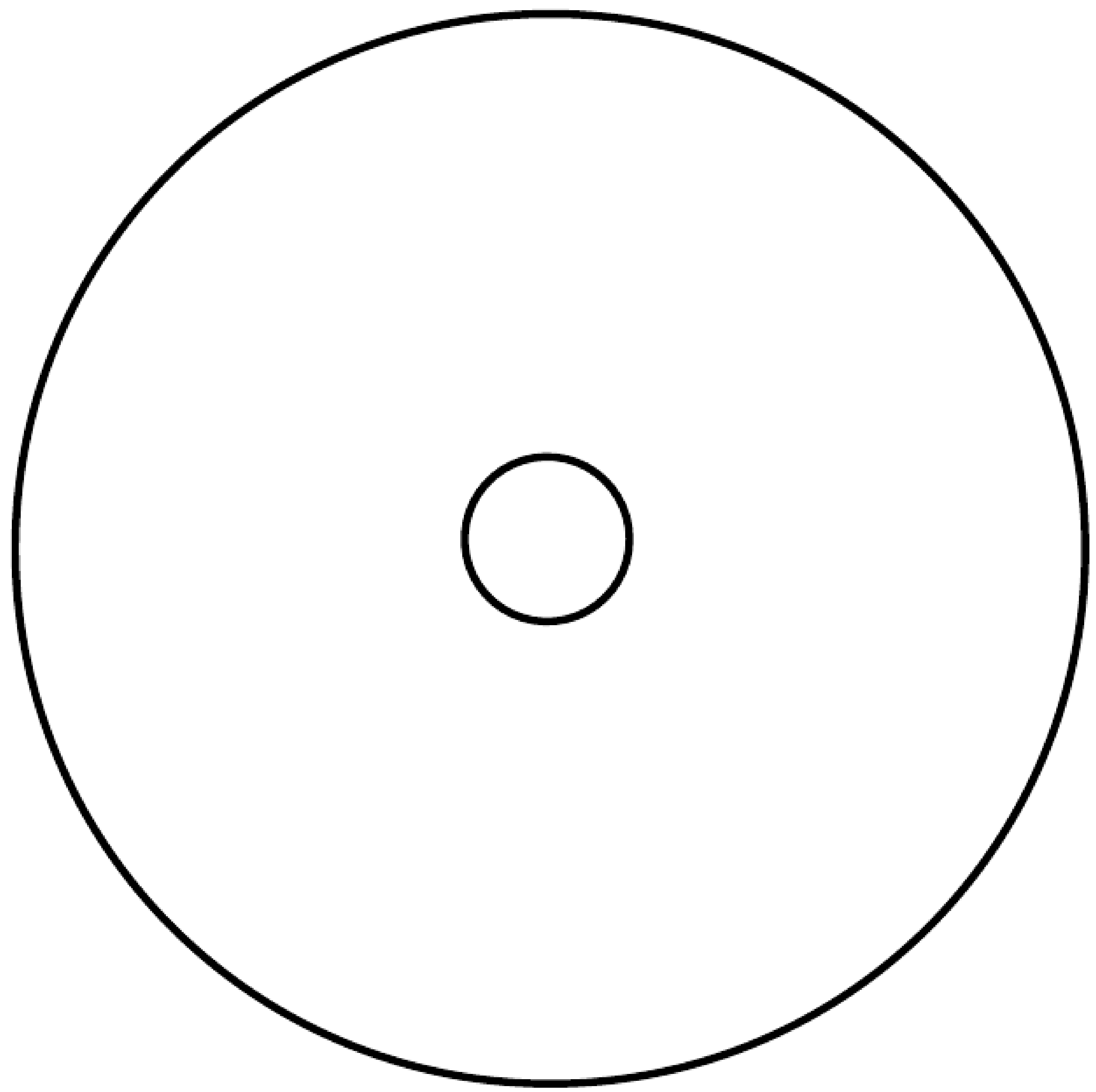
by M. S. NARASIMHAN

Canonical hermitian metrics

We now introduce a hermitian metric on M. To do this it is sufficient to introduce a positive definite hermitian form on $H^1(X, W(\mathrm{Ad}\rho))$. Since $W(\mathrm{Ad}\rho)$ is given by a local system, the operator of exterior differentiation, d, is well defined on C^∞ differential forms with values in $W(\mathrm{Ad}\rho)$. Let $T(\rho)$ denote the space of d-closed C^∞ forms of type $(0, 1)$ with coefficient in $W(\mathrm{Ad}\rho)$. Then $T(\rho)$ is canonically isomorphic to $H^1(X, W(\mathrm{Ad}\rho))$. So it suffices to introduce a positive definite hermitian form on $T(\rho)$. If $\omega \in T(\rho)$, let $\omega^\#$ denote the $(1, 0)$ form with coefficients in $W(\mathrm{Ad}\rho)$ obtained by using the conjugation $A \mapsto A^*$ in $\mathfrak{gl}(n, C)$. (A^* denotes the conjugate transpose of A. Locally, if $\omega = A(z)\,d\bar{z}$, $\omega^\# = A^*(z)\,dz$). Define the hermitian scalar product in $T(\rho)$ by

$$(\omega_1, \omega_2) = \frac{1}{i} \int_X \mathrm{Trace}\,(\omega_1, \omega_2^\#),\ \omega_1, \omega_2 \in T(\rho),$$

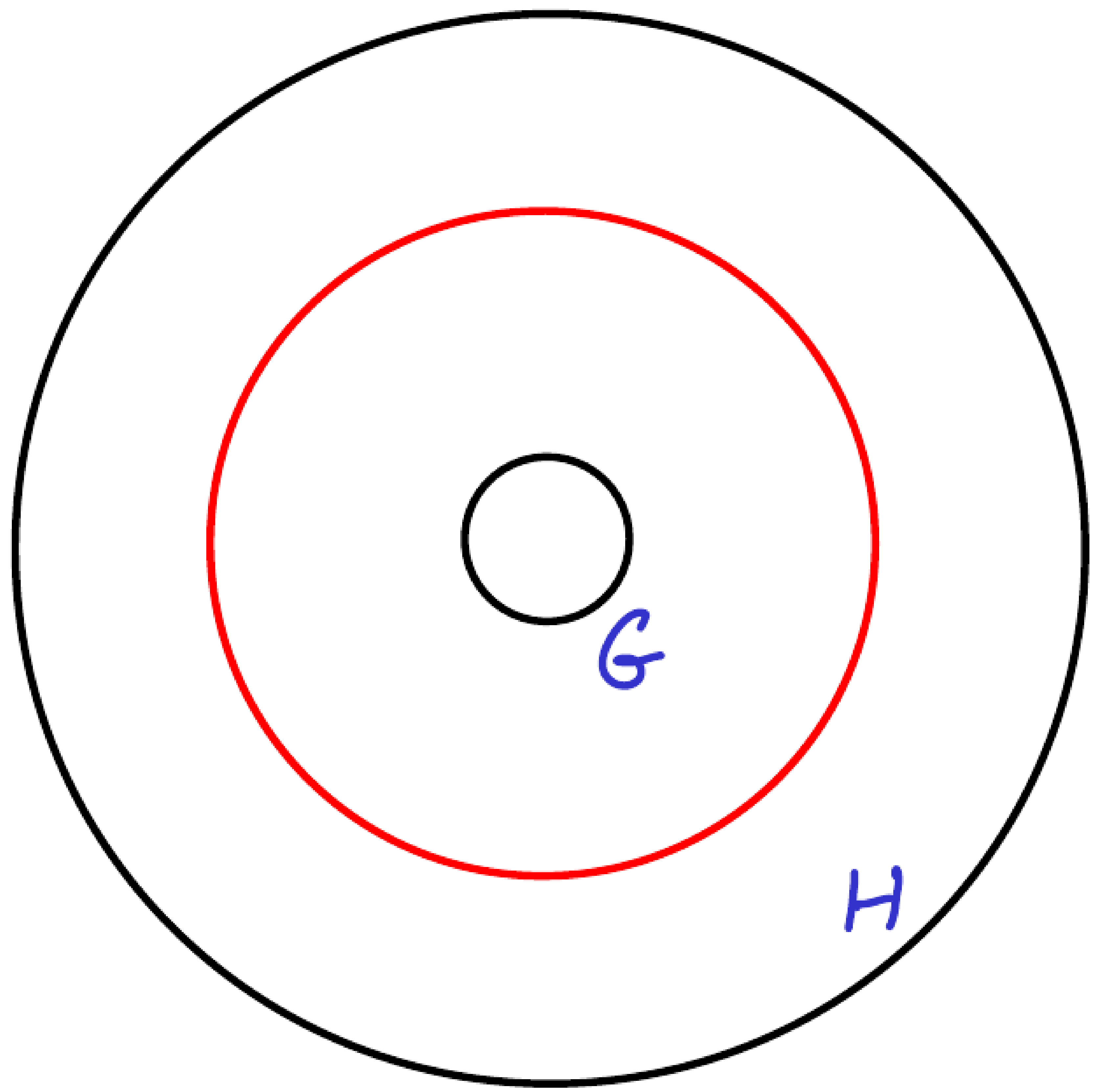
Fission spaces



Fission spaces

$V = \bigoplus_{i \in I} V_i$ I graded vector space

$G = GL(V) \supset H = GrAut(V) \cong \prod GL(V_i)$

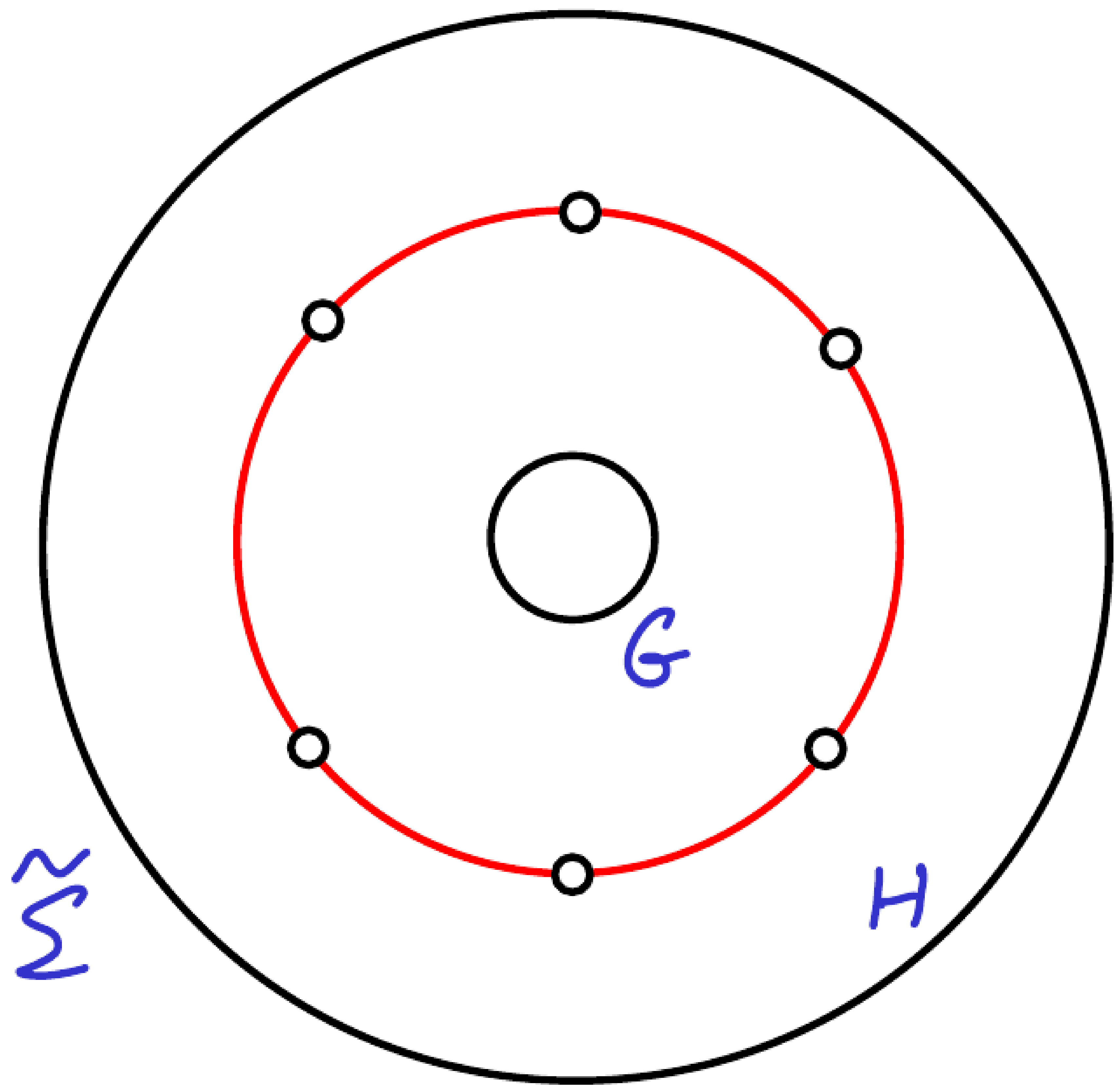


Fission spaces

$V = \bigoplus_{i \in I} V_i$ $\quad$ I graded vector space

$G = GL(V) \supset H = \mathrm{GrAut}(V) \cong \prod GL(V_i)$

- $2k$ tangential punctures $\circ$



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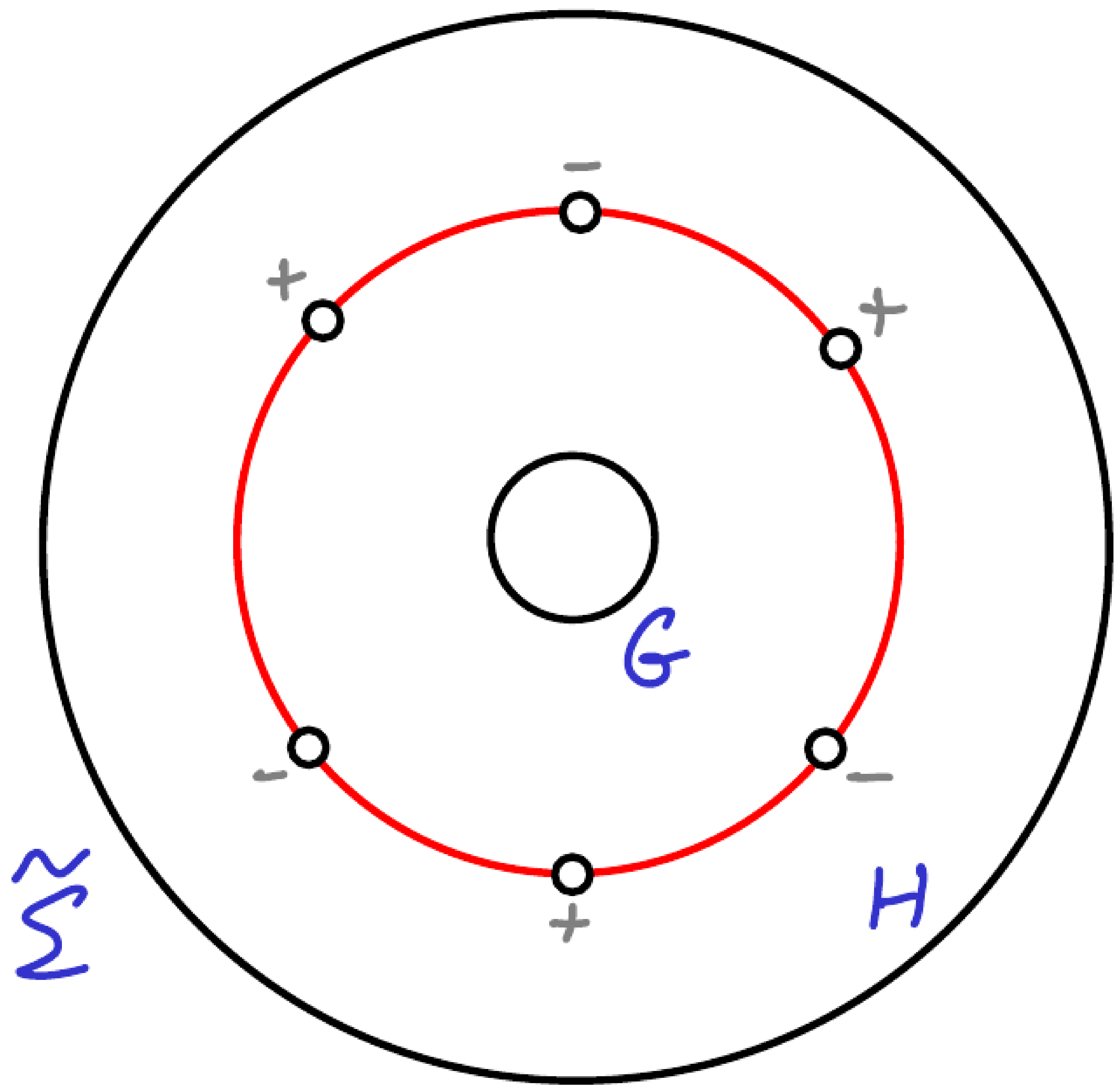

- $2k$ tangential punctures ○
- $U_+, U_- = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \subset G$ (Stokes groups)

Fission spaces

$V = \bigoplus_{i \in I} V_i$ I graded vector space

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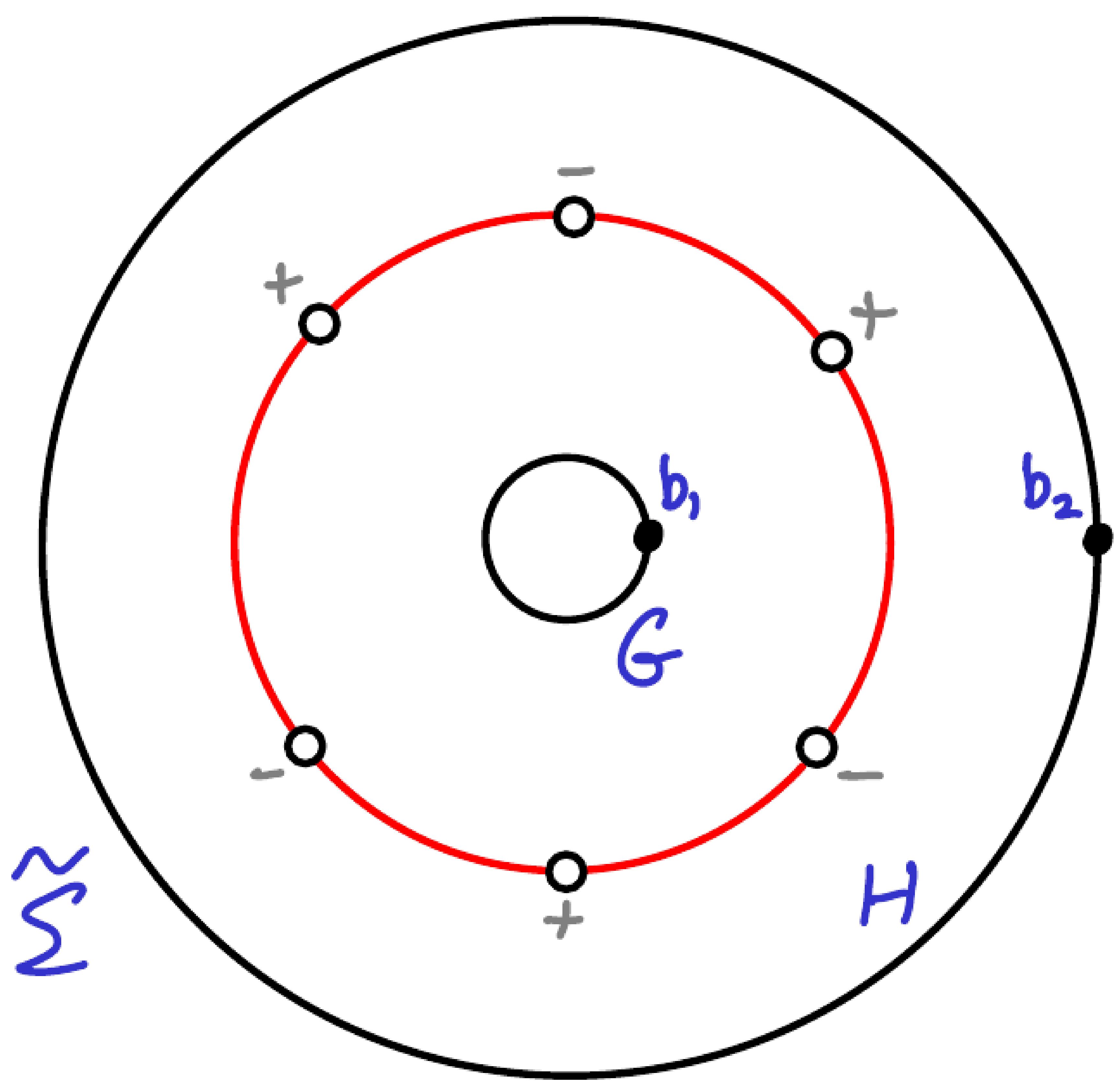

- $2k$ tangential punctures $\circ$
- $U_+, U_- = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \subset G$ (Stokes groups)
- $\Pi = \Pi_1(\tilde{\Sigma}, \{b_1, b_2\})$ (wild surface groupoid)

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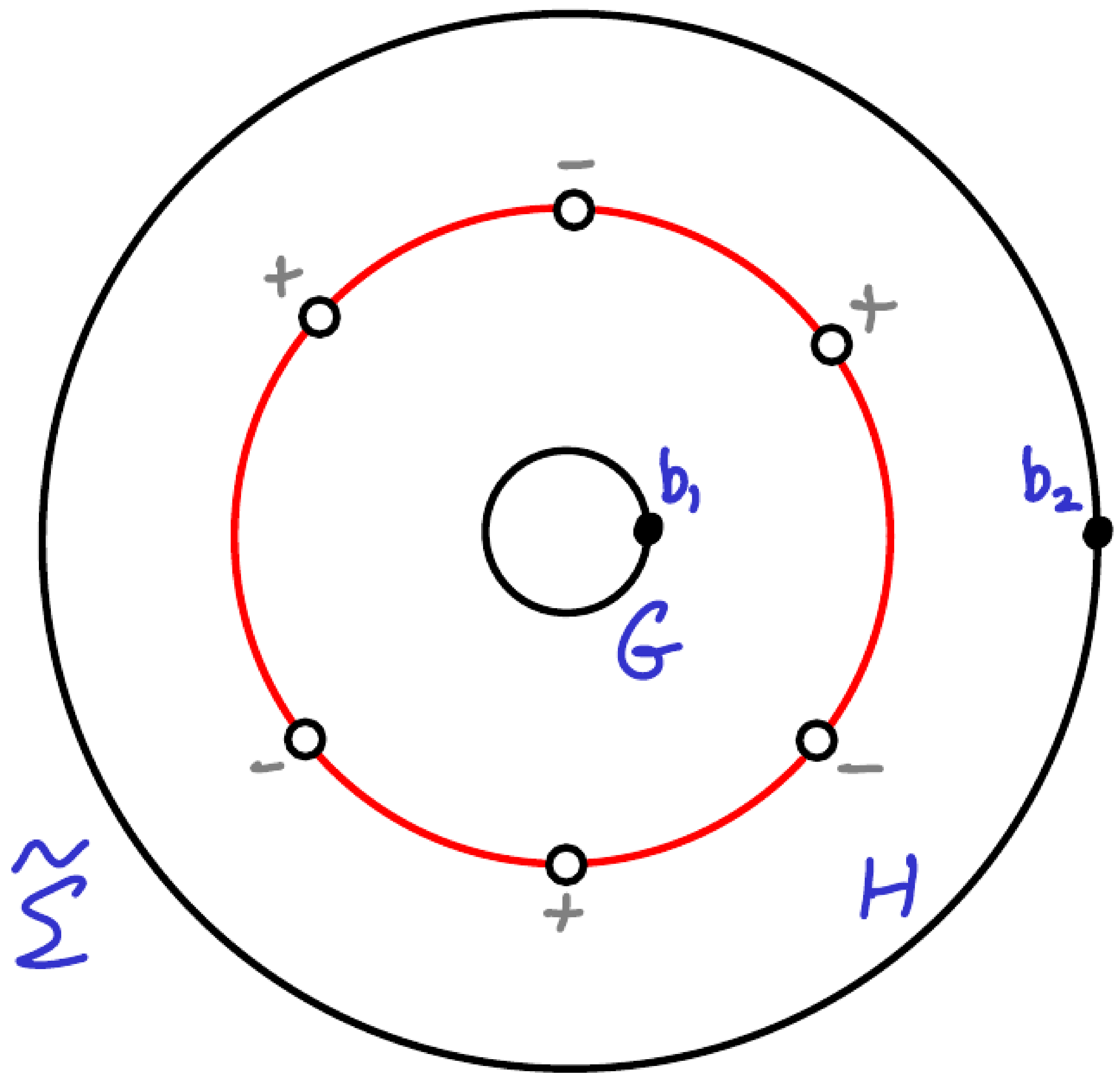

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- $\Pi = \Pi_1(\tilde{\Sigma}, \{b_1, b_2\})$ (wild surface groupoid)
- $\mathcal{A} = {}_G\mathcal{A}^k_H = \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$

$$\cong G \times H \times (U_+ \times U_-)^k$$

$$\cong \{\text{Stokes local systems framed at } b_1, b_2\}/\text{iso.}$$

Fission spaces

$V = \bigoplus_{i\in I} V_i$ $\quad$ I graded vector space

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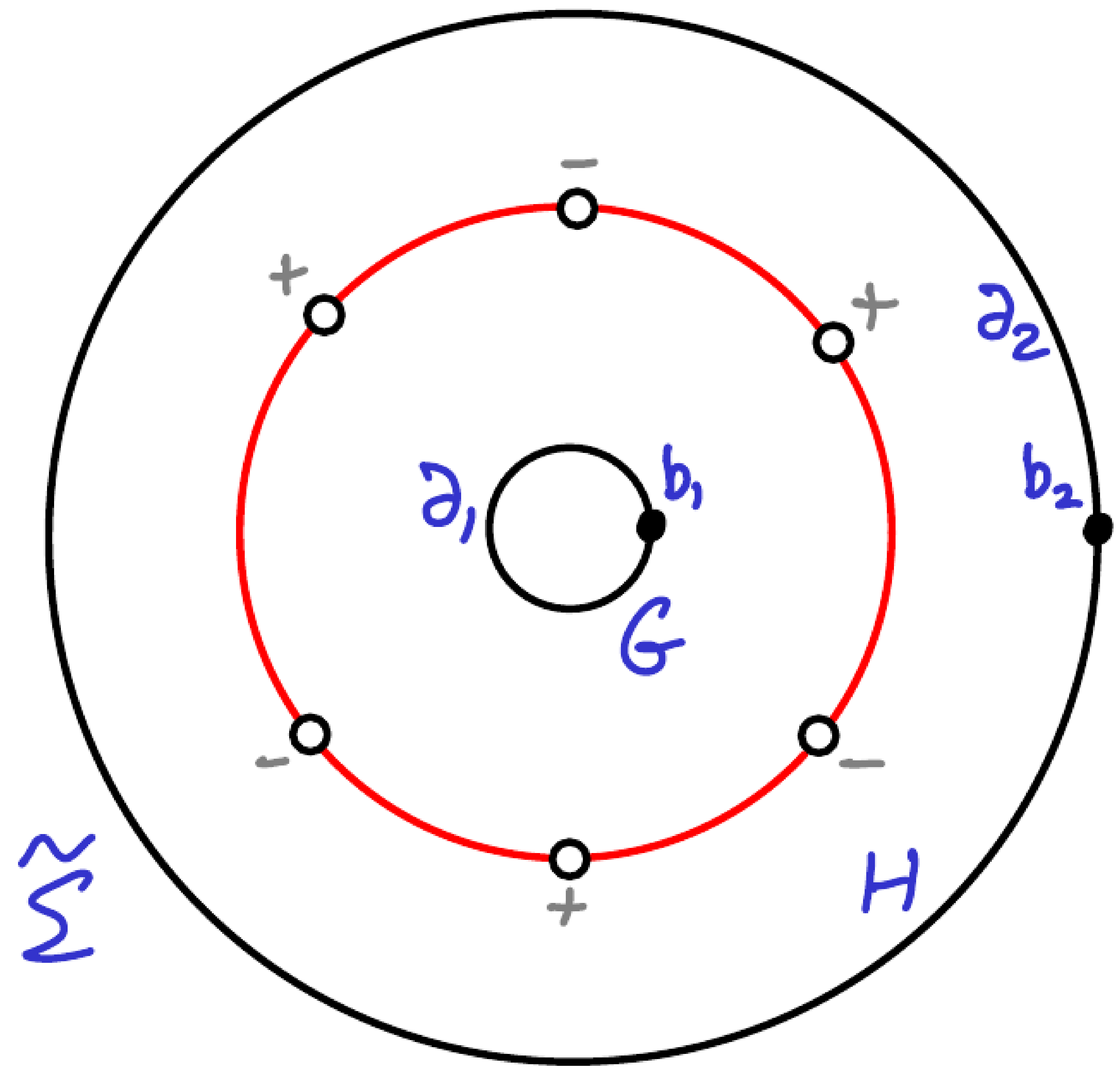

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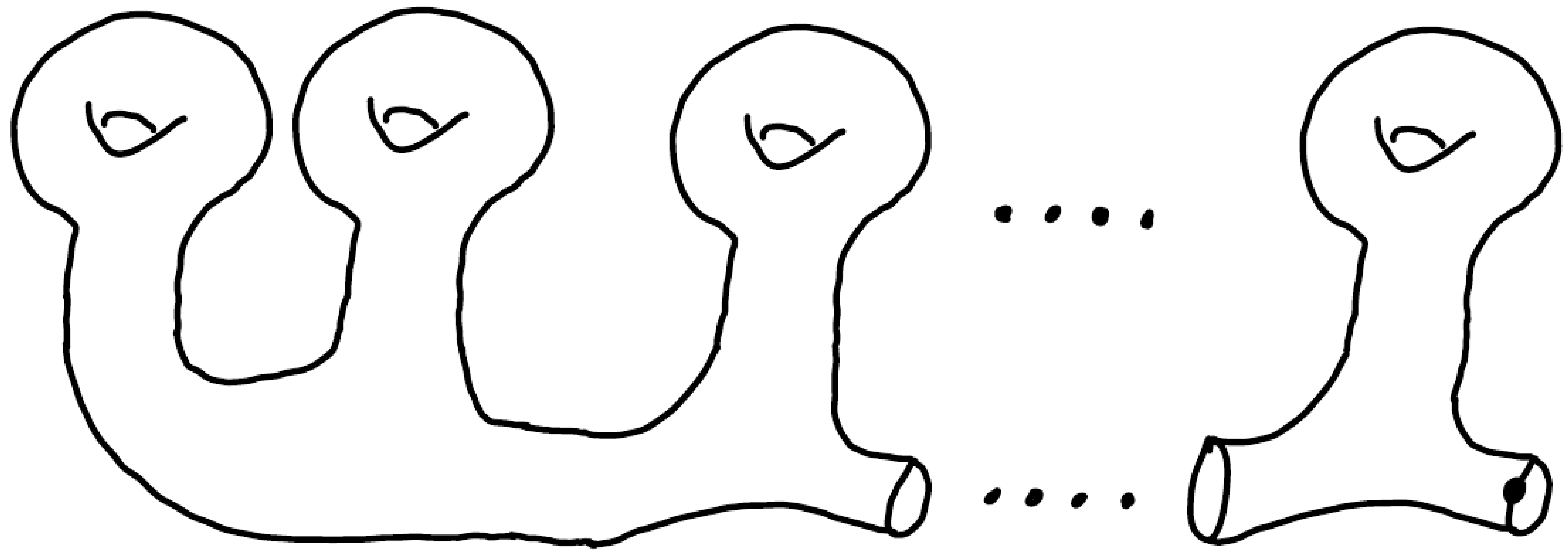
  $\cong G \times H \times (U_+ \times U_-)^k$

  $\cong \{\text{Stokes local systems framed at } b_1, b_2\}/\text{iso.}$

Thm $\mathcal{A}$ is a quasi-Hamiltonian $G \times H$ space with moment map $\mu : \mathcal{A} \to G \times H$, $\quad \mu(\rho) = (\rho(\partial_1), \rho(\partial_2))$

(2002 $H = T$ (any G), 2009 any H, G ($k=1$), 2011 in general)

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)

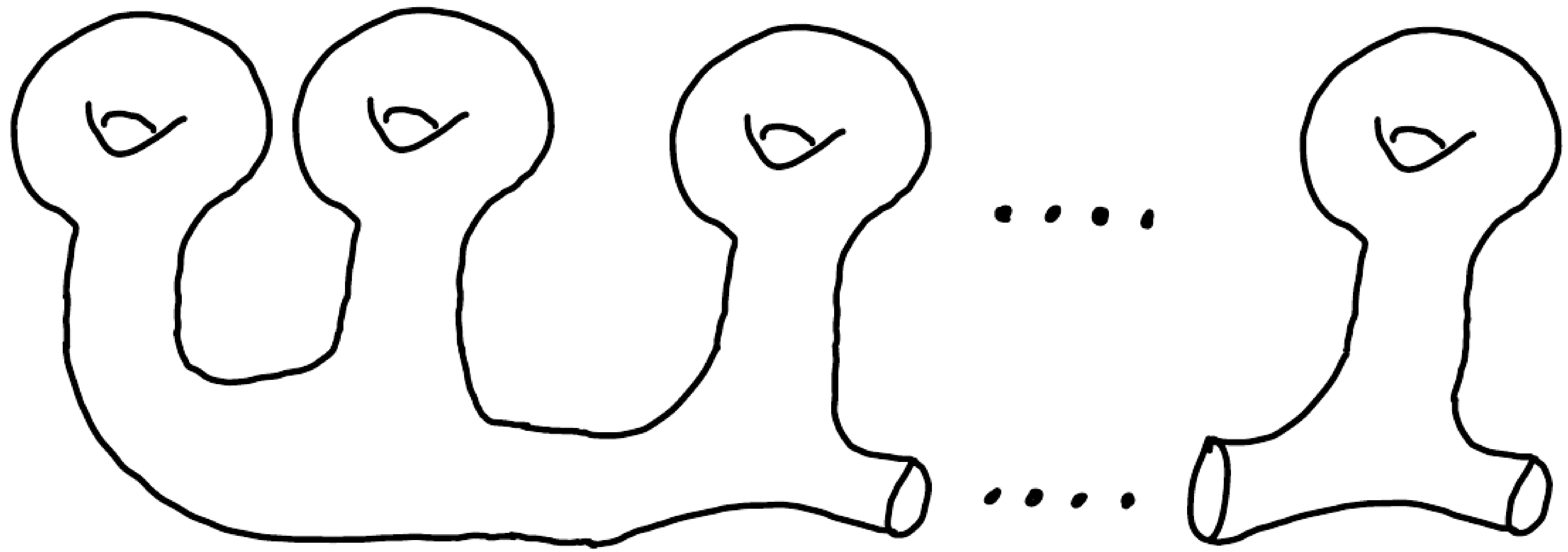


Thm. $R = \mathrm{Hom}(\pi_1(\Sigma_{g,1}), G)$ is a quasi-Hamiltonian G-space

$\cong G^{2g}$, $\mu = [A_1, B_1] \cdots [A_g, B_g] : R \to G$

$[a,b] = aba^{-1}b^{-1}$

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)



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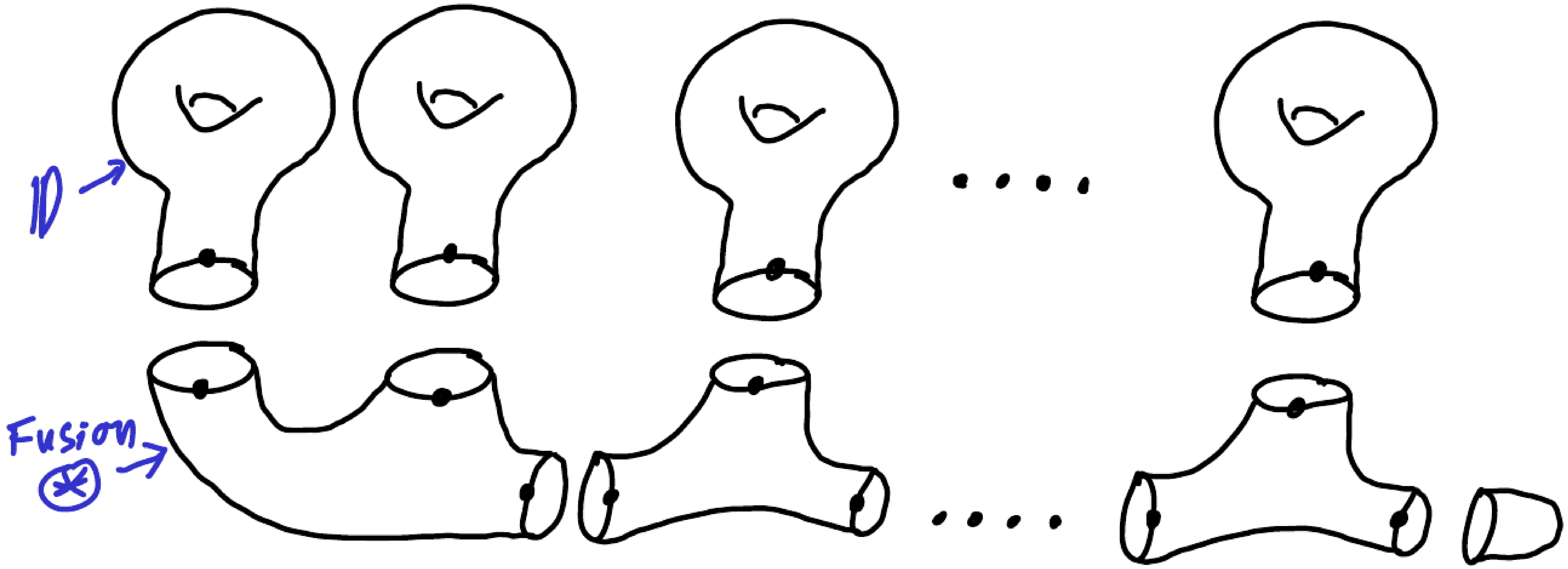
$[a,b] = aba^{-1}b^{-1}$

Cor.
- $\mathcal{M}_B = R/G$ is a Poisson variety
- The symplectic leaves are $\mathcal{M}_B(\mathcal{C}) = \mu^{-1}(\mathcal{C})/G$ for conjugacy classes $\mathcal{C} \subset G$

E.g. $\mathcal{M}_B(\Sigma_g) = R/\!/G = \mu^{-1}(1)/G = \{A, B \in G^{2g} \mid \prod [A_i, B_i] = 1\}/G$

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)



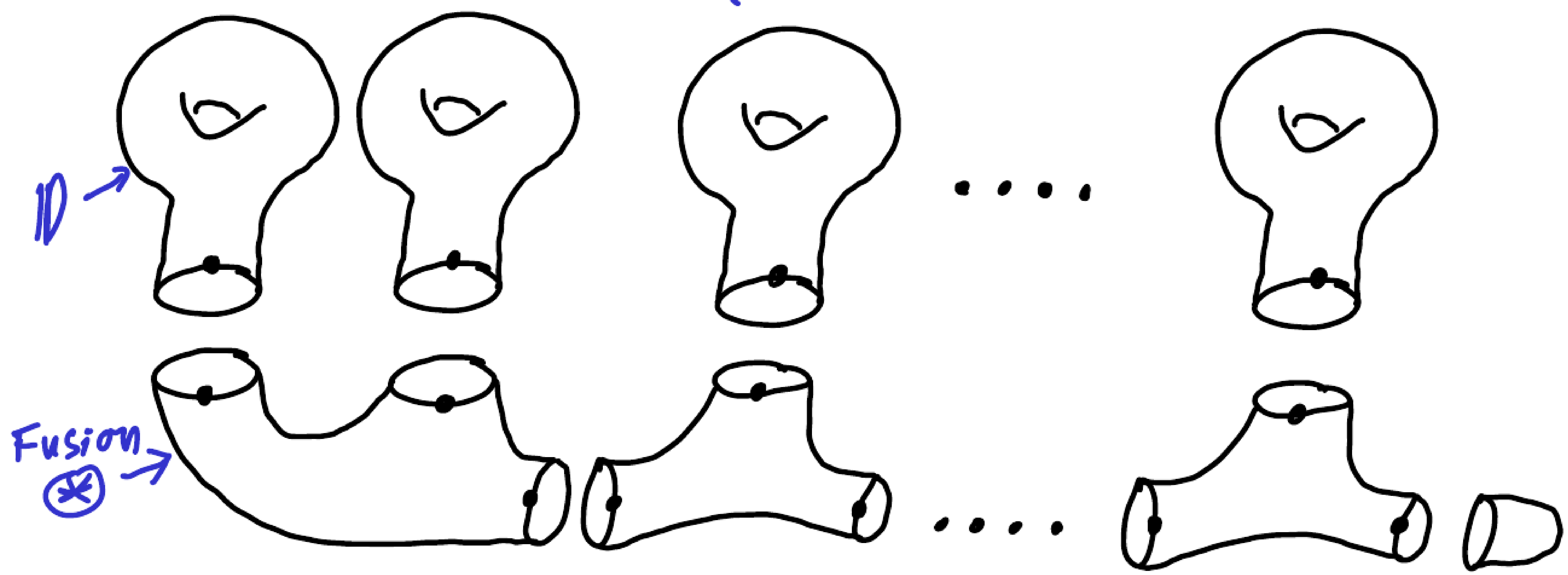
Thm. $\mathcal{R} = \mathrm{Hom}(\pi_1(\Sigma_{g,1}), G)$ is a quasi-Hamiltonian G-space

$$\cong G^{2g}, \quad \mu = [A_1, B_1] \cdots [A_g, B_g] : \mathcal{R} \to G$$

$[a,b] = aba^{-1}b^{-1}$

Cor.
- $\mathcal{M}_B = \mathcal{R}/G$ is a Poisson variety
- The symplectic leaves are $\mathcal{M}_B(\mathcal{C}) = \mu^{-1}(\mathcal{C})/G$ for conjugacy classes $\mathcal{C} \subset G$
- Can fuse simple pieces: $\mathcal{R} = \mathbb{D} \circledast \cdots \circledast \mathbb{D}$, $\mathbb{D} = \mathcal{R}(\Sigma_{1,1})$

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)

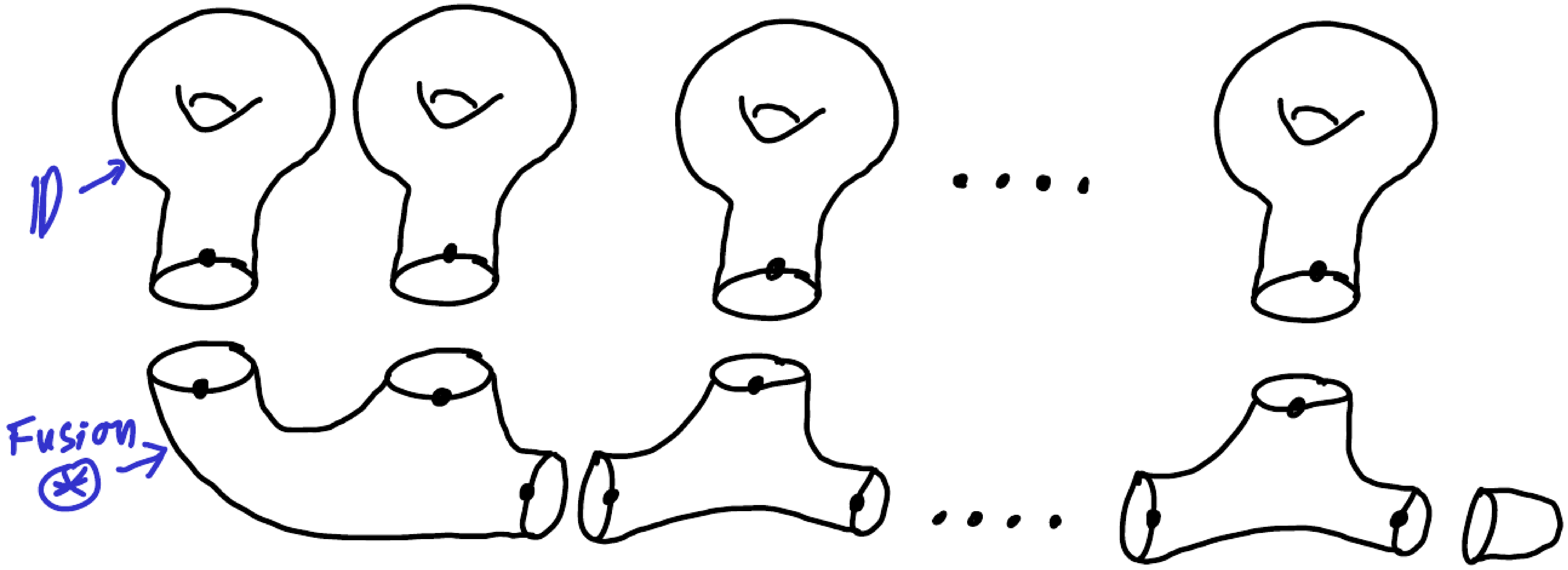


Toolbox:

- $\mathbb{D} = \mathcal{R}(\Sigma_{1,1}) \cong G \times G$, • $\mathcal{C} \subset G$
- $D = \mathcal{R}(\Sigma_{0,2}) = \mathcal{R}(\quad) \cong G \times G$ "double"
- $\circledast$ fusion , • $\oslash$ reduction $(/\!\!/ G)$

$$\mathcal{M}_B(\underset{\sim}{\mathcal{C}}) = \mathbb{D} \circledast \cdots \circledast \mathbb{D} \circledast \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!\!/ G$$

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)



Toolbox: • $\mathbb{D} = \mathcal{R}(\Sigma_{1,1}) \cong G\times G$, • $\mathcal{C} \subset G$

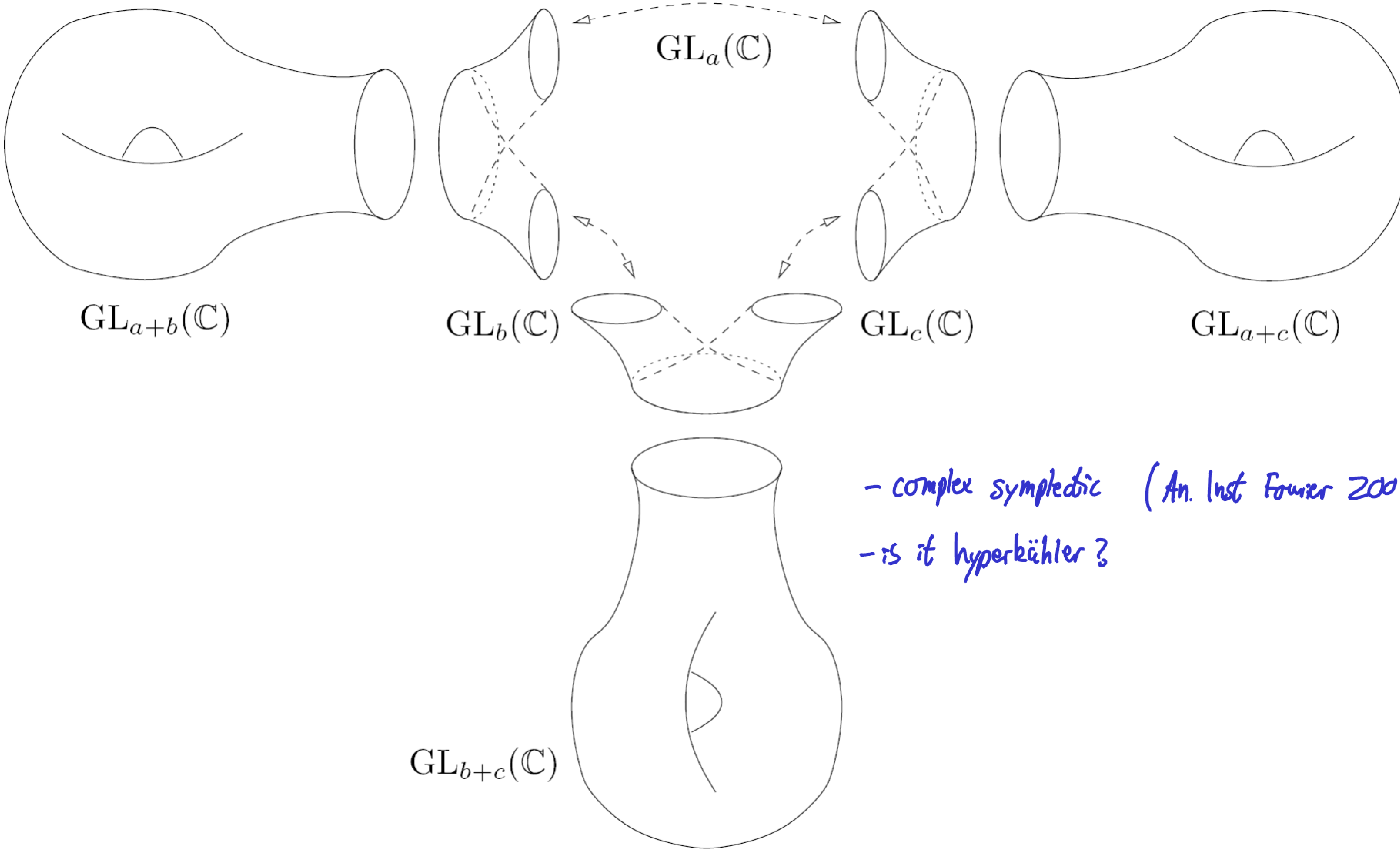
• $D = \mathcal{R}(\Sigma_{0,2}) = \mathcal{R}(\quad) \cong G \times G$ "double"

• ⊛ fusion , • ⦶ reduction ($/\!\!/G$)

Now add fission spaces $\mathcal{A} = {}_G\mathcal{A}_H^k$ $\forall\, G, H, k$

$\Rightarrow$ lots of new algebraic symplectic/Poisson varieties

"fission varieties" $\supsetneq$ (untwisted) wild character varieties



- complex symplectic (An. Inst Fourier 2009)
- is it hyperkähler?

Wild character varieties

E.g. Birkhoff 1913 wrote presentations in generic setting:

$$(C_1^{-1} h_1 S_{2k_1}^{(1)} \cdots S_1^{(1)} C_1) \cdots (C_m^{-1} h_m S_{2k_m}^{(m)} \cdots S_1^{(m)} C_m) = 1$$

(see Jimbo-Miwa-Ueno 1981 equation 2.46)

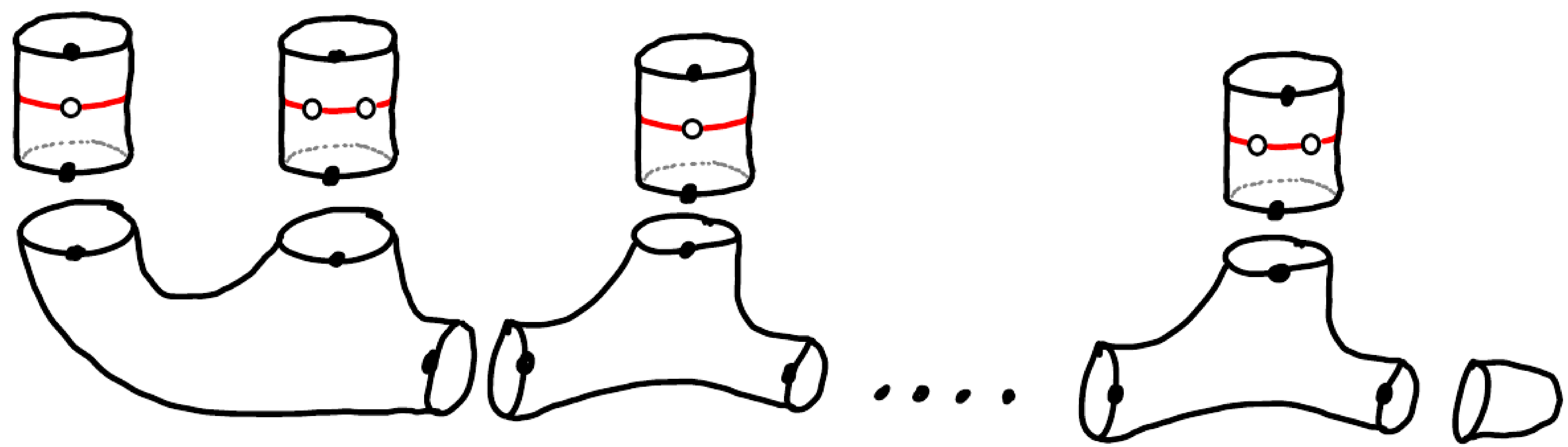
Wild character varieties

E.g. Birkhoff 1913 wrote presentations in generic setting:

$$(C_1^{-1}h_1 S^{(1)}_{2k_1}\cdots S^{(1)}_1 C_1)\cdots(C_m^{-1}h_m S^{(m)}_{2k_m}\cdots S^{(m)}_1 C_m)=1$$

(see Jimbo-Miwa-Ueno 1981 equation 2.46)

$$R = {}_G\mathcal{A}_T^{k_1} \underset{G}{\circledast} {}_G\mathcal{A}_T^{k_2} \underset{G}{\circledast} \cdots \underset{G}{\circledast} {}_G\mathcal{A}_T^{k_m} \xrightarrow{\mu} T^m \times G$$



<u>Thm</u> Reductions with fixed $h_i \in T$ are symplectic

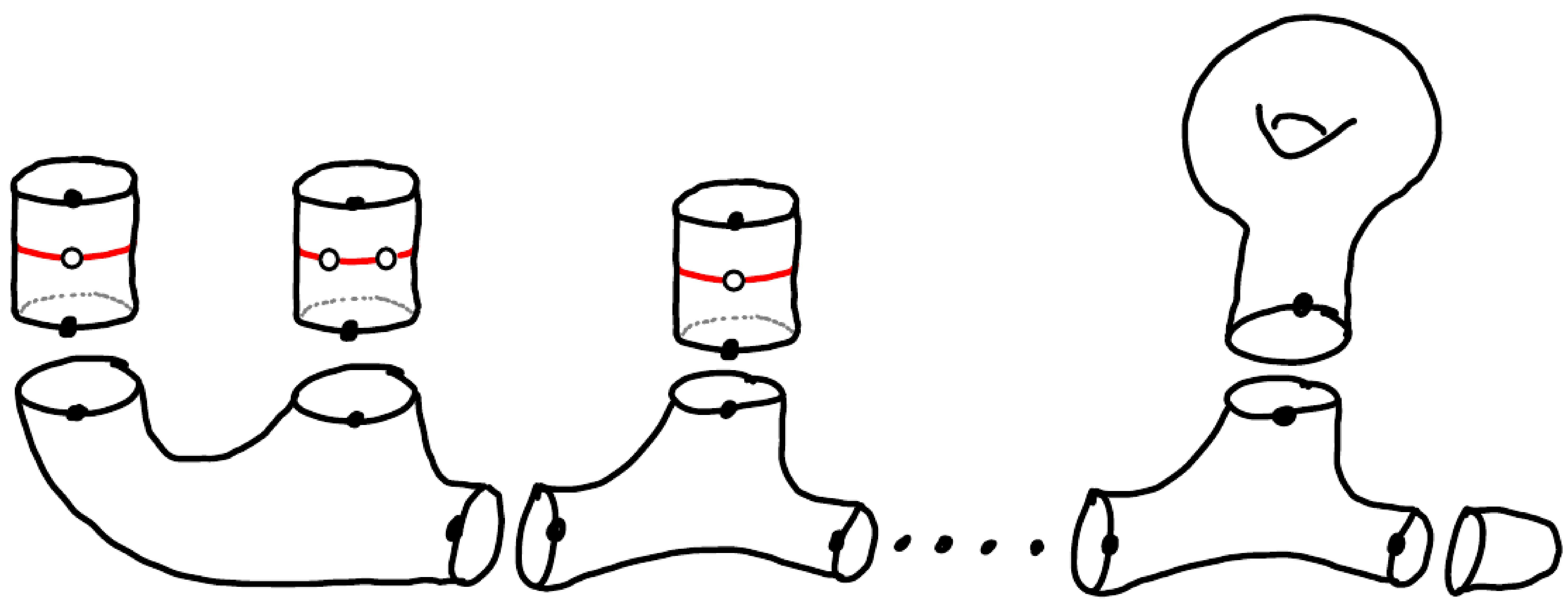
(Adv. Math. 2001 "irreg. Atiyah Bott", algebraic quasi-Hamiltonian approach 2002)

Wild character varieties

Similarly in general ($\sim$ any alg. connections on twisted G-bundles)

$$(C_1^{-1}h_1 S_{k_1}^{(1)}\cdots S_1^{(1)}C_1)\cdots(C_m^{-1}h_m S_{k_m}^{(m)}\cdots S_1^{(m)}C_m)\overrightarrow{\prod_1^g} A_iB_iA_i^{-1}B_i^{-1} = 1$$

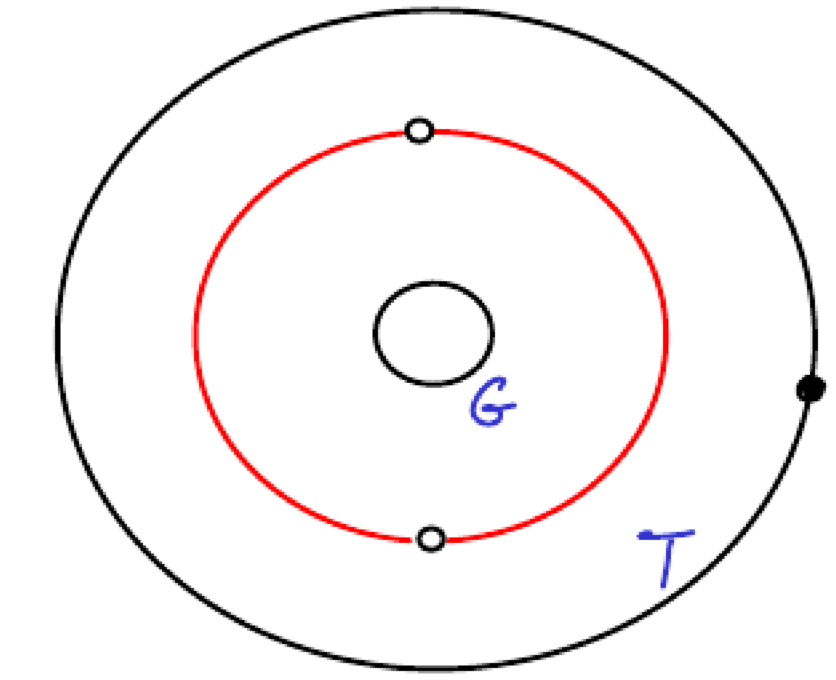
$$\mathrm{THom}_{\mathbb{S}}(\Pi, G) = \mathcal{A}_1 \circledast \cdots \circledast \mathcal{A}_m \circledast \mathbb{D}^{\circledast g} /\!/ G$$



Thm Wild character variety $\mathcal{M}_B = \mathrm{THom}_{\mathbb{S}}(\Pi, G)/\underset{\sim}{H}$ is a Poisson variety with symplectic leaves got by fixing (twisted) conjugacy classes of formal monodromy

... An. Inst. Fourier '09, arXiv:1111.6228, arXiv:1512.08091 (with D. Yamakawa)

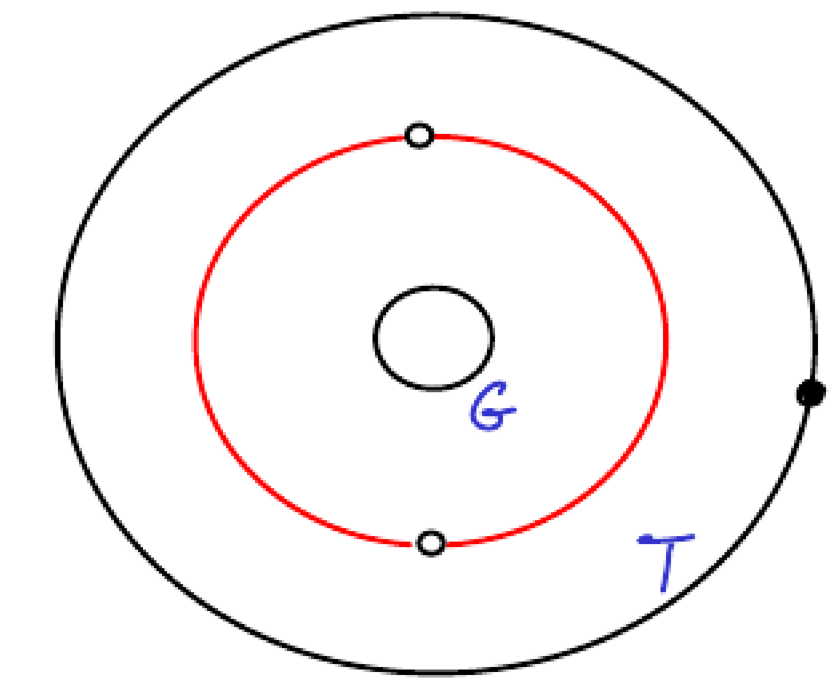
Wild character varieties



E.g. $\quad {}_G\mathcal{A}'_T/G \cong T \times U_+ \times U_-$

is thus a nonlinear Poisson variety (with Hamiltonian T-action)

Wild character varieties



E.g. $\quad {}_G\mathcal{A}'_T/G \cong T \times U_+ \times U_-$

is thus a nonlinear Poisson variety (with Hamiltonian T-action)

Thm (Drinfeld / Semenov Tian Shansky, DeConcini Procesi 1993)

$U_q(\mathfrak{g})$ quantizes a Poisson variety $G^* \cong T \times U_+ \times U_-$

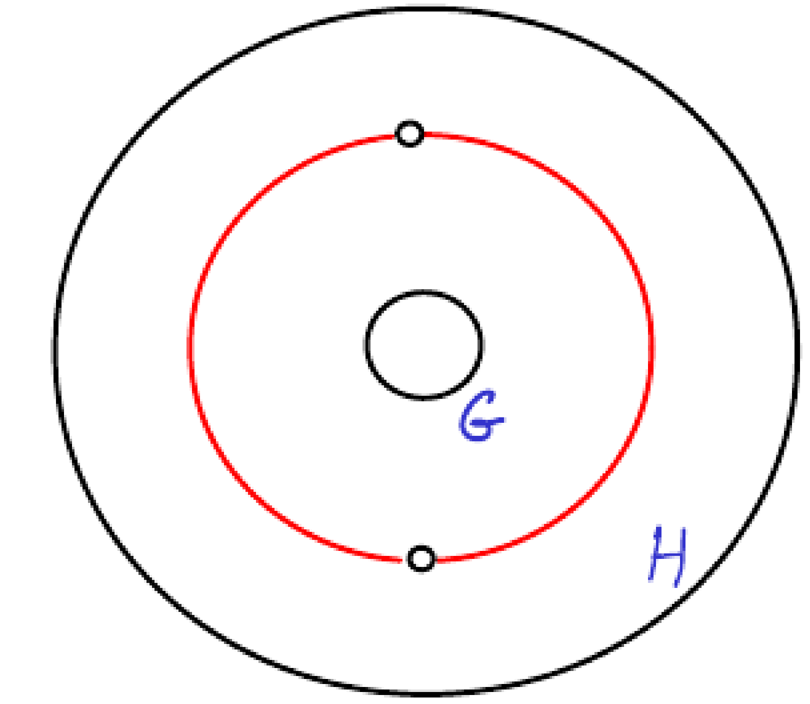
Thm (PB Invent. Math 2001)

$G^* \cong {}_G\mathcal{A}'_T/G$ as a Poisson variety

Cor. The Drinfeld-Jimbo quantum group is modular

(comes from moduli of connections on curves)

Wild character varieties



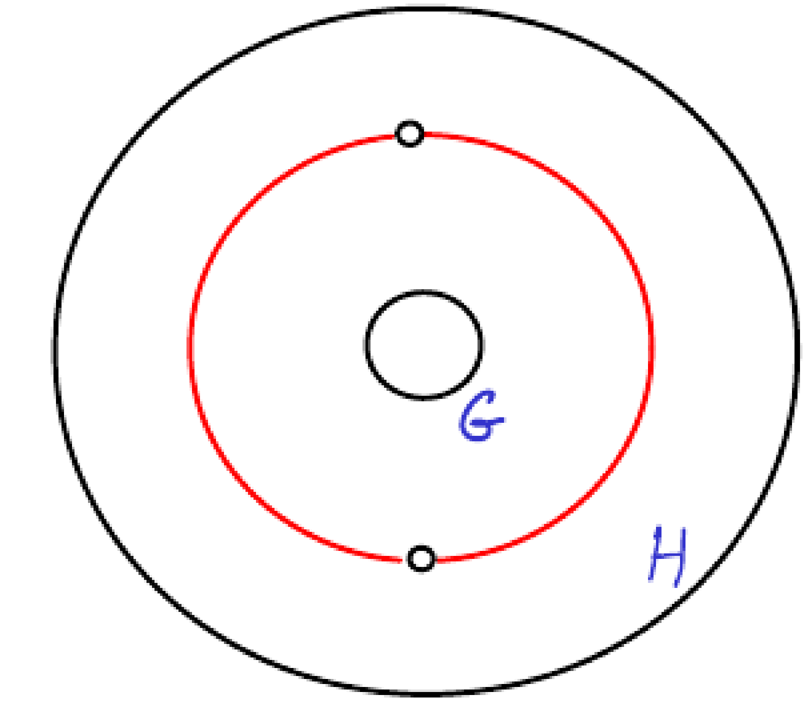
E.g. $\quad {}_G\mathcal{A}^1_H / G\times H \cong (H \times U_+ \times U_-)/H$

is an algebraic Poisson variety with symplectic leaves

$$\mathcal{M}_B(\mathcal{C}, \check{\mathcal{C}}) = \{ h, S_1, S_2 \mid h \in \check{\mathcal{C}}, \ h S_1 S_2 \in \mathcal{C} \}/H$$

for conjugacy classes $\check{\mathcal{C}} \subset H$, $\mathcal{C} \subset G$

Wild character varieties



E.g. $\quad {}_G\mathcal{A}^{\prime}_H / G\times H \cong (H \times U_+ \times U_-)/H$

is an algebraic Poisson variety with symplectic leaves

$$\mathcal{M}_B(\mathcal{C}, \check{\mathcal{C}}) = \{ h, S_1, S_2 \mid h \in \check{\mathcal{C}},\ h S_1 S_2 \in \mathcal{C} \}/H$$

for conjugacy classes $\check{\mathcal{C}} \subset H$, $\mathcal{C} \subset G$

Thm (Fourier-Laplace, Malgrange 1991)

This class of varieties $\cong$ all tame genus zero character varieties

Thm – symplectic structures match too (PB arXiv 1307)

– and the hyperkähler metrics (Sz. Szabo arXiv 1407)

$\rightsquigarrow$ notion of "representations" of abstract moduli space

Sakai's question

Plato to Painlevé (McKay–Harnad) c.f. PB 0706.2634 Exercise 3

Sakai's question

Plato to Painlevé (McKay–Harnad) c.f. PB 0706.2634
Exercise 3

groups: Tetra. Octa. Icosa. $\subset SO_3(\mathbb{R})$

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$$\uparrow$$

binary groups: $\tilde{T}$ $\tilde{O}$ $\tilde{I}$ $\subset$ $SU_2 \subset SL_2(\mathbb{C})$

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$\uparrow$ $\uparrow$ $\uparrow$

resolve: X_T X_O X_I

Sakai's question

Plato to Painlevé (McKay–Harnad) c.f. PB 0706.2634 Exercise 3

groups:	Tetra.	Octa.	Icosa.	$\subset$	$SO_3(\mathbb{R})$
					$\uparrow$
binary groups:	$\tilde{T}$	$\tilde{O}$	$\tilde{I}$	$\subset$	$SU_2 \subset SL_2(\mathbb{C})$
singularities:	$\mathbb{C}^2/\tilde{T}$	$\mathbb{C}^2/\tilde{O}$	$\mathbb{C}^2/\tilde{I}$		
	$\uparrow$	$\uparrow$	$\uparrow$		
resolve: + deform	$\underset{\sim}{X}_T$	$\underset{\sim}{X}_O$	$\underset{\sim}{X}_I$		
	$\downarrow$	$\downarrow$	$\downarrow$		
	$\mathbb{C}^6$	$\mathbb{C}^7$	$\mathbb{C}^8$		

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	$\uparrow$	$\uparrow$	$\uparrow$	
resolve:	X_T	X_O	X_I	
+ deform	$\tilde{}$ $\downarrow$	$\tilde{}$ $\downarrow$	$\tilde{}$ $\downarrow$	
	$\mathbb{C}^6$	$\mathbb{C}^7$	$\mathbb{C}^8$	
	$\circlearrowleft$	$\circlearrowleft$	$\circlearrowleft$	
Weyl groups:	$W(E_6)$	$W(E_7)$	$W(E_8)$	

Sakai's question

Plato to Painlevé (McKay–Harnad) c.f. PB 0706.2634 Exercise 3

groups: Tetra. Octa. Icosa. $\subset SO_3(\mathbb{R})$

$$\uparrow$$

binary groups: $\tilde{T}$ $\tilde{O}$ $\tilde{I}$ $\subset SU_2 \subset SL_2(\mathbb{C})$

singularities: $\mathbb{C}^2/\tilde{T}$ $\mathbb{C}^2/\tilde{O}$ $\mathbb{C}^2/\tilde{I}$

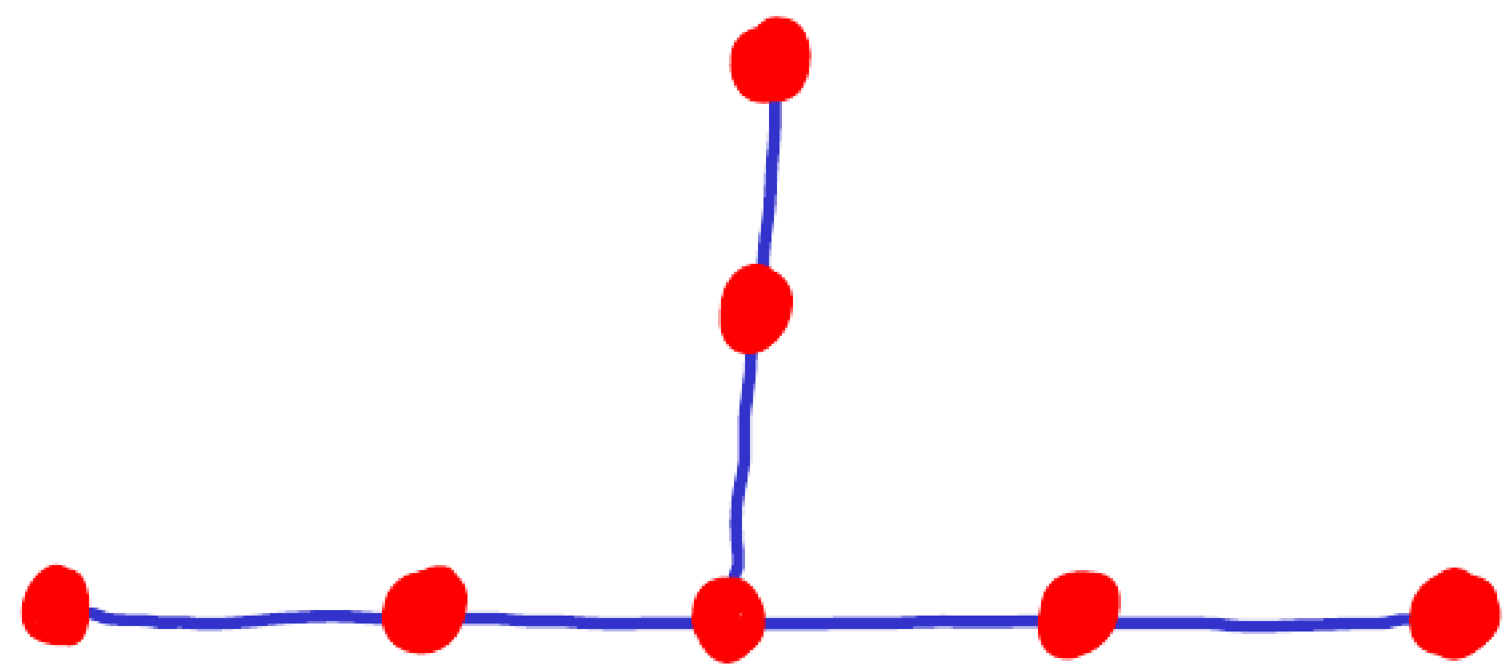
resolve + deform: $\uparrow$ $\tilde{X}_T$ $\downarrow$ $\mathbb{C}^6$; $\uparrow$ $\tilde{X}_O$ $\downarrow$ $\mathbb{C}^7$; $\uparrow$ $\tilde{X}_I$ $\downarrow$ $\mathbb{C}^8$

Weyl groups: $\circlearrowleft W(E_6)$ $\circlearrowleft W(E_7)$ $\circlearrowleft W(E_8)$

Kronheimer (1989):

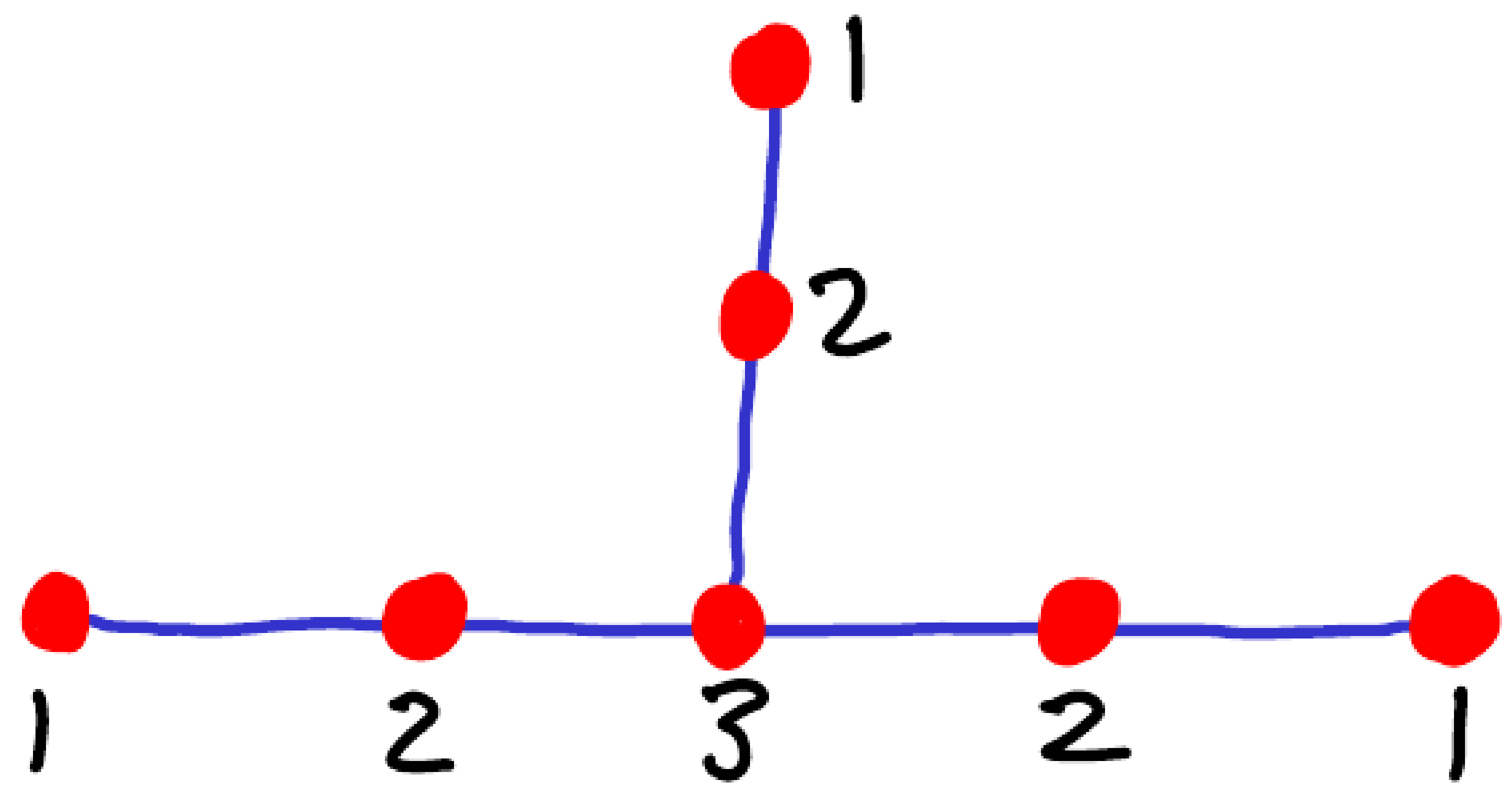
- smooth fibres are complete hyperkähler 4-folds
- construct in terms of affine Dynkin graph

E.g. E_6 case (hol. symplectic approach)



E.g. E_6 case (hol. symplectic approach)

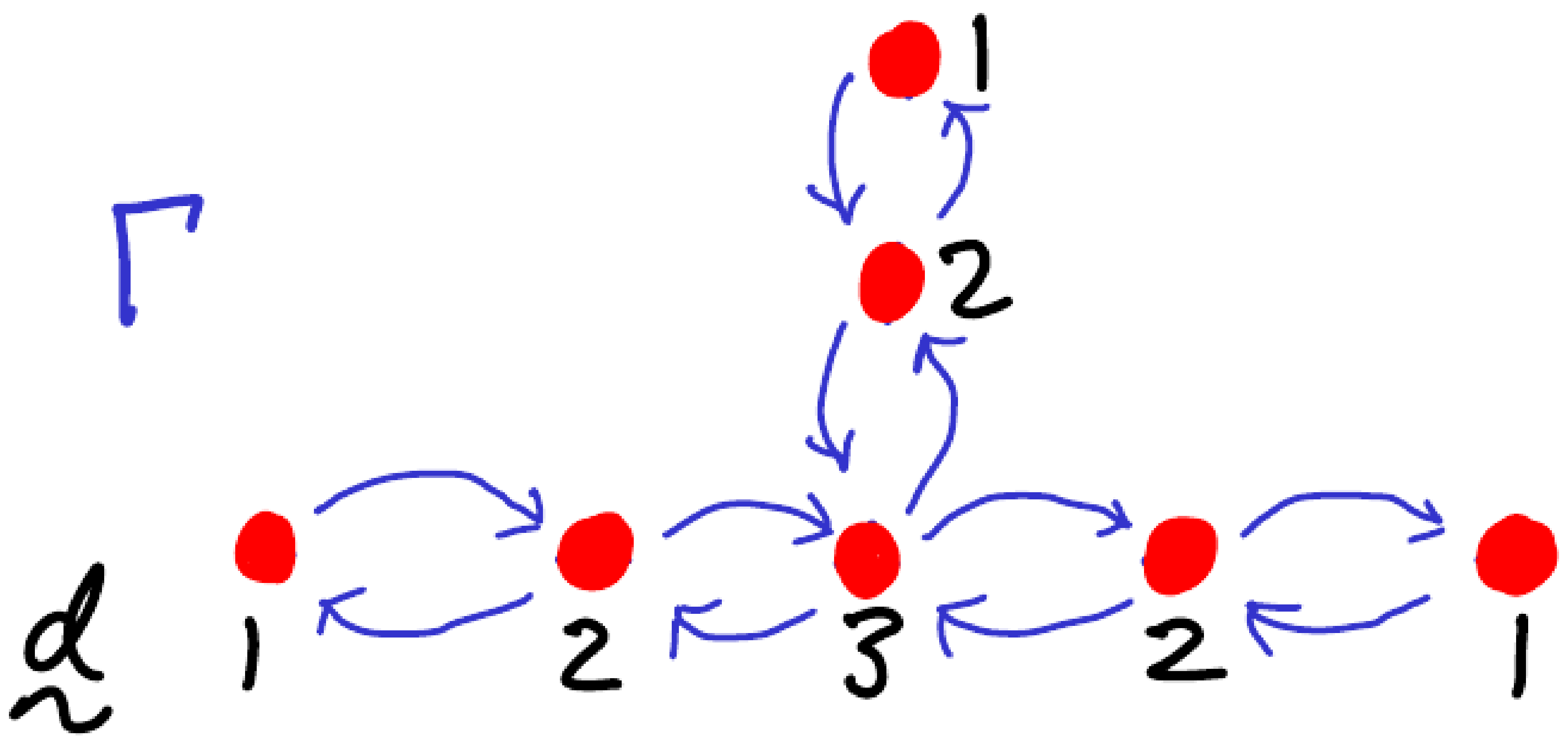
Γ



$\underset{\sim}{d}$

$$\mathbb{V} = \mathrm{Rep}(\Gamma, \mathbb{C}^{\underset{\sim}{d}})$$

E.g. E_6 case (hol. symplectic approach)



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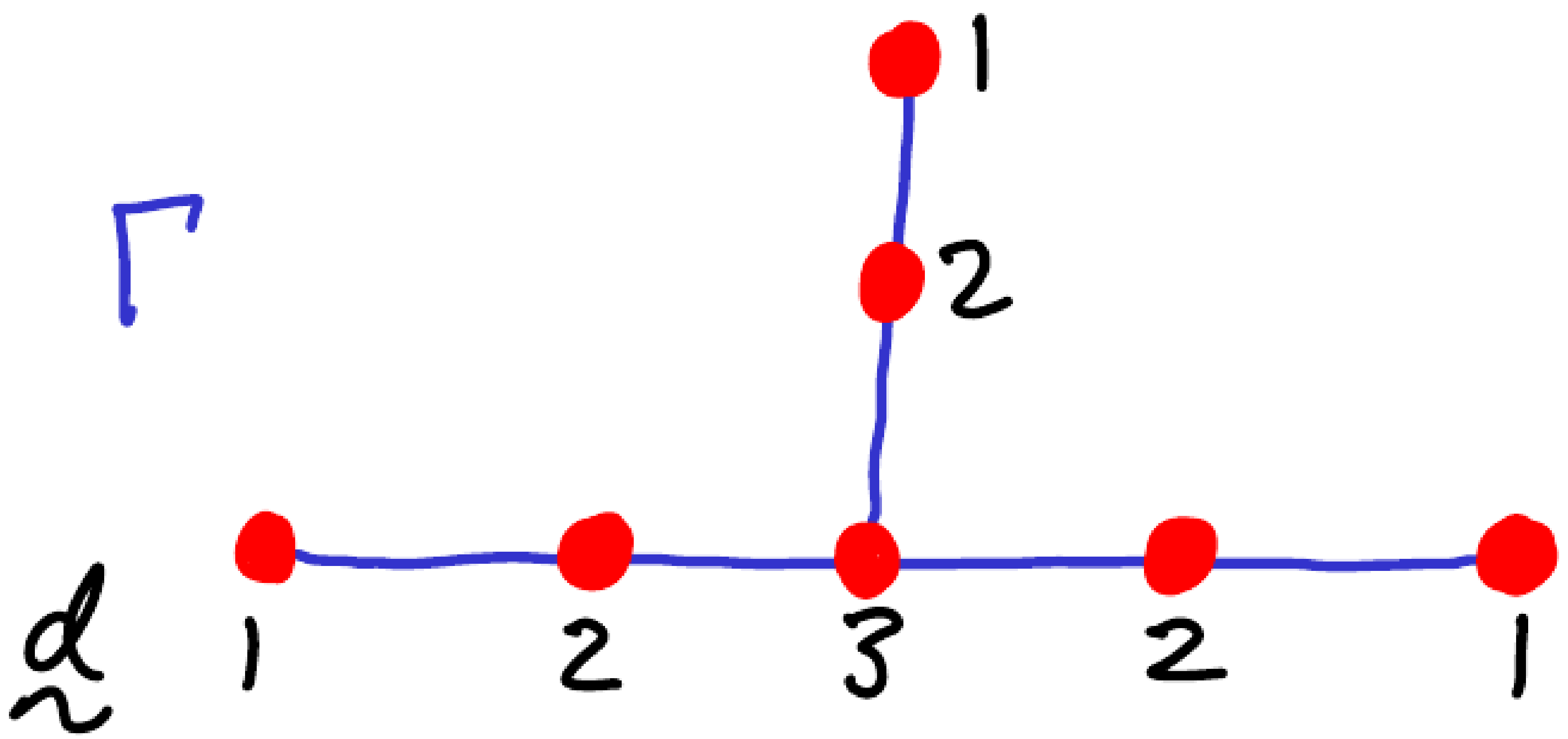

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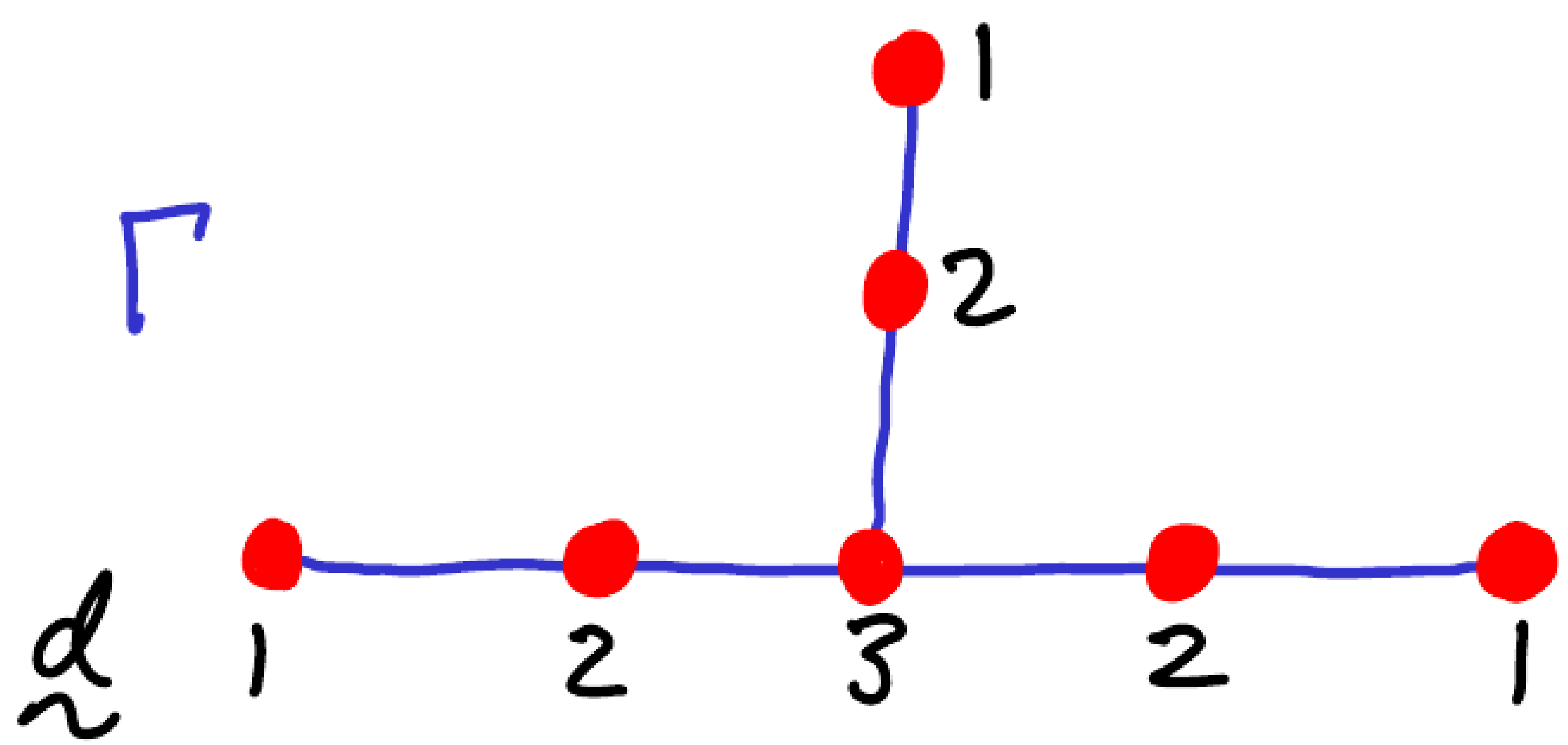
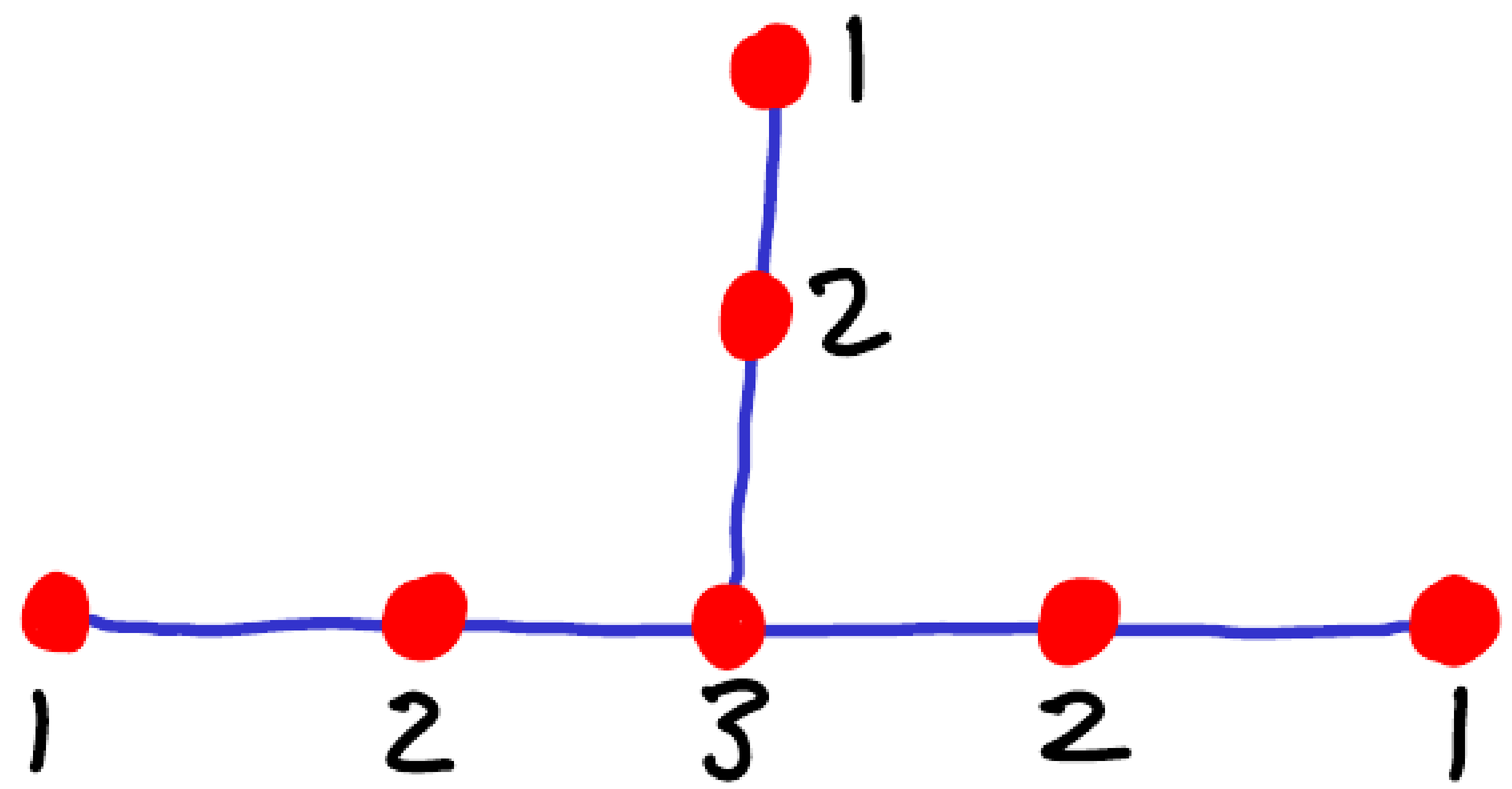


$$\mathbb{V} = \mathrm{Rep}(\Gamma, \mathbb{C}^{\underset{\sim}{d}})$$

$$\mathbb{G} = GL(\mathbb{C}^{\underset{\sim}{d}}) = \prod GL_{d_i}(\mathbb{C})$$

E.g. E_6 case (hol. symplectic approach)

Γ



$\underset{\sim}{d}$

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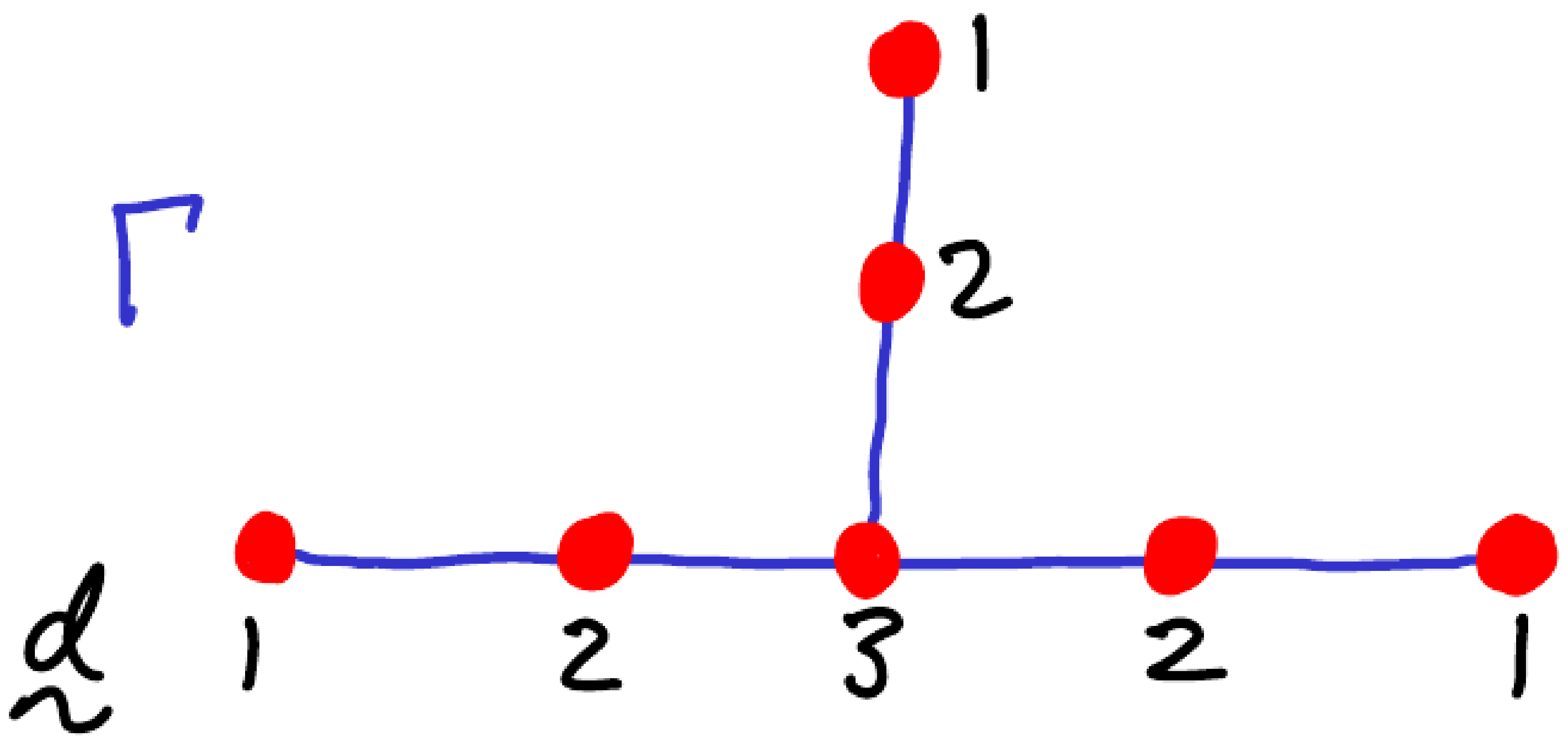
$$\mathbb{G} = GL(\mathbb{C}^{\underset{\sim}{d}}) = \prod GL_{d_i}(\mathbb{C})$$

$$N = NQV(\Gamma, \underset{\sim}{\lambda}, \underset{\sim}{d}) = \mathbb{V} /\!\!/_{\underset{\sim}{\lambda}} \mathbb{G} = \mu^{-1}(\underset{\sim}{\lambda}) / \mathbb{G}$$

$$\underset{\sim}{\lambda} \in \mathrm{Lie}(\mathbb{G})^* \cong \prod \mathrm{End}(\mathbb{C}^{d_i}) \quad \text{central}$$

E.g. E_6 case (hol. symplectic approach)

Γ



$$\boxed{\dim_{\mathbb{C}}(N) = 2 - (\underset{\sim}{d}, \underset{\sim}{d})}$$

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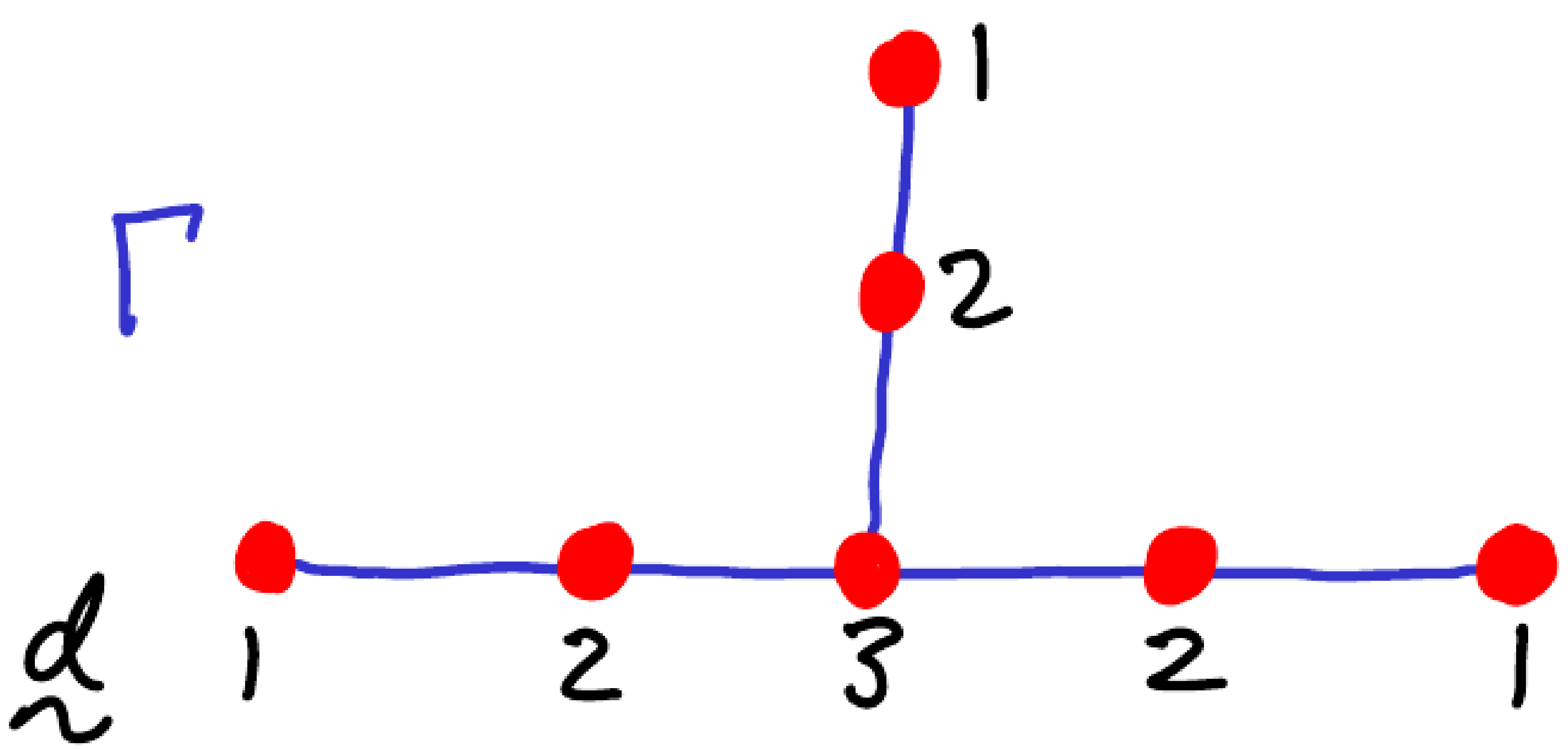

$$\boxed{\dim_{\mathbb{C}}(N) = 2 - (\underset{\sim}{d}, \underset{\sim}{d})}$$

N is <u>modular</u> $\cong$ moduli space of Fuchsian systems $\mathcal{M}^*$

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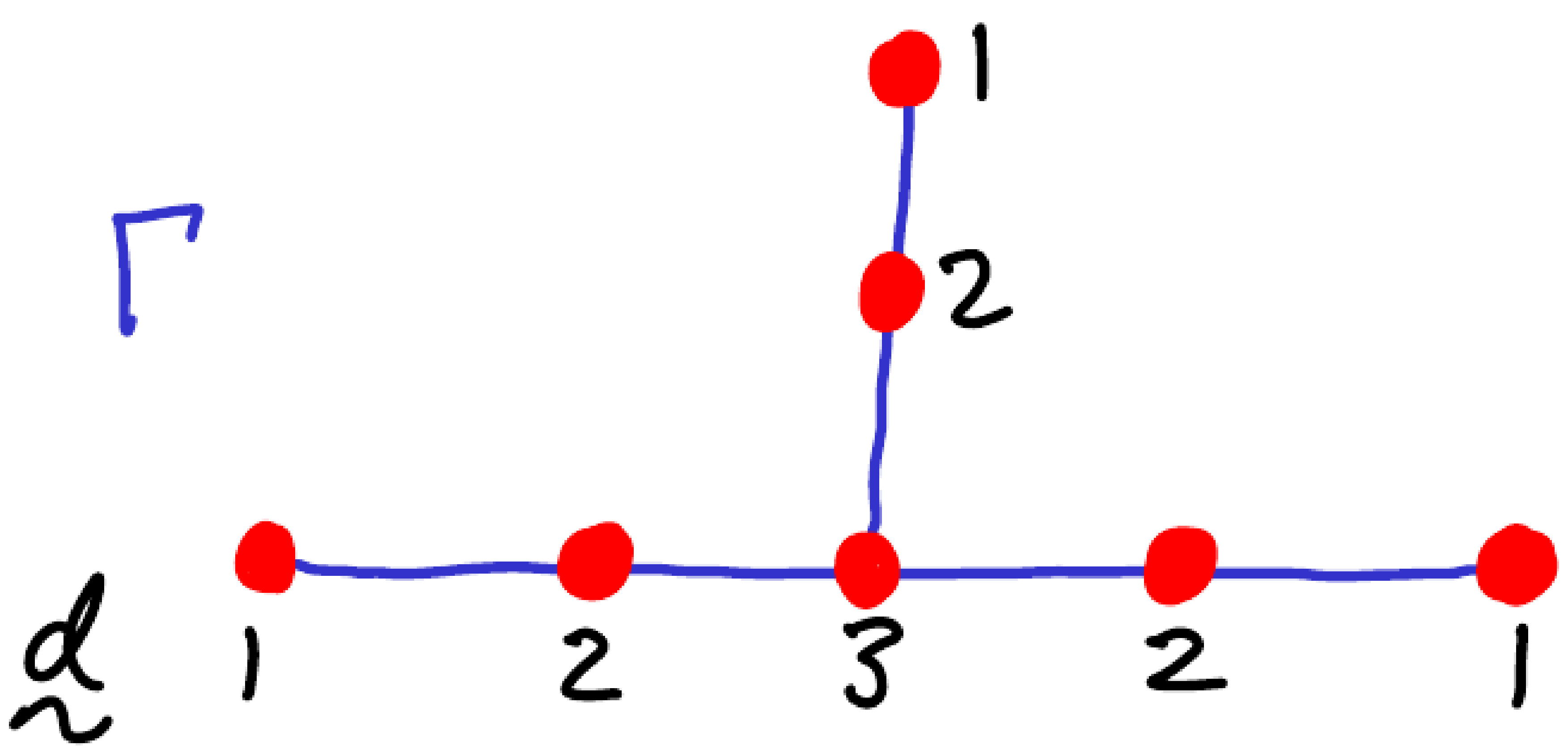


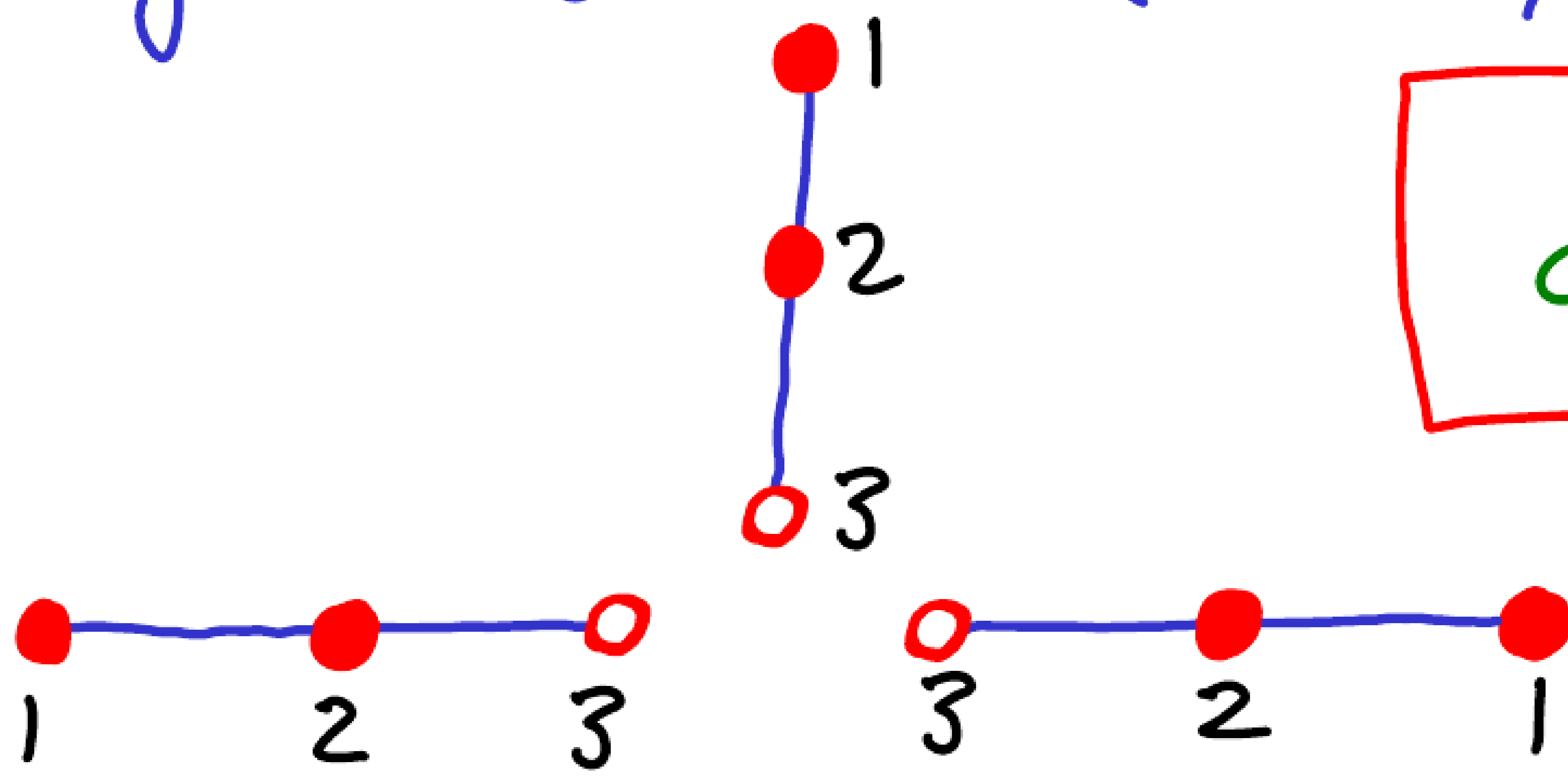
$$\boxed{\dim_{\mathbb{C}}(\mathcal{N}) = 2 - (\underset{\sim}{d}, \underset{\sim}{d})}$$

$\mathcal{N}$ is <u>modular</u> $\cong$ moduli space of Fuchsian systems $\mathcal{M}^*$

$$\cong \mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3 /\!\!/ GL_3(\mathbb{C}) \qquad \left(\begin{array}{c}\mathcal{O}_i \subset \mathfrak{gl}_3(\mathbb{C}) \\ \text{coadjoint orbit} \\ \text{dim } 6\end{array}\right)$$

$$\dim_{\mathbb{C}} = 6 + 6 + 6 - 2(9-1) = 2$$

E.g. E_6 case (hol. symplectic approach)



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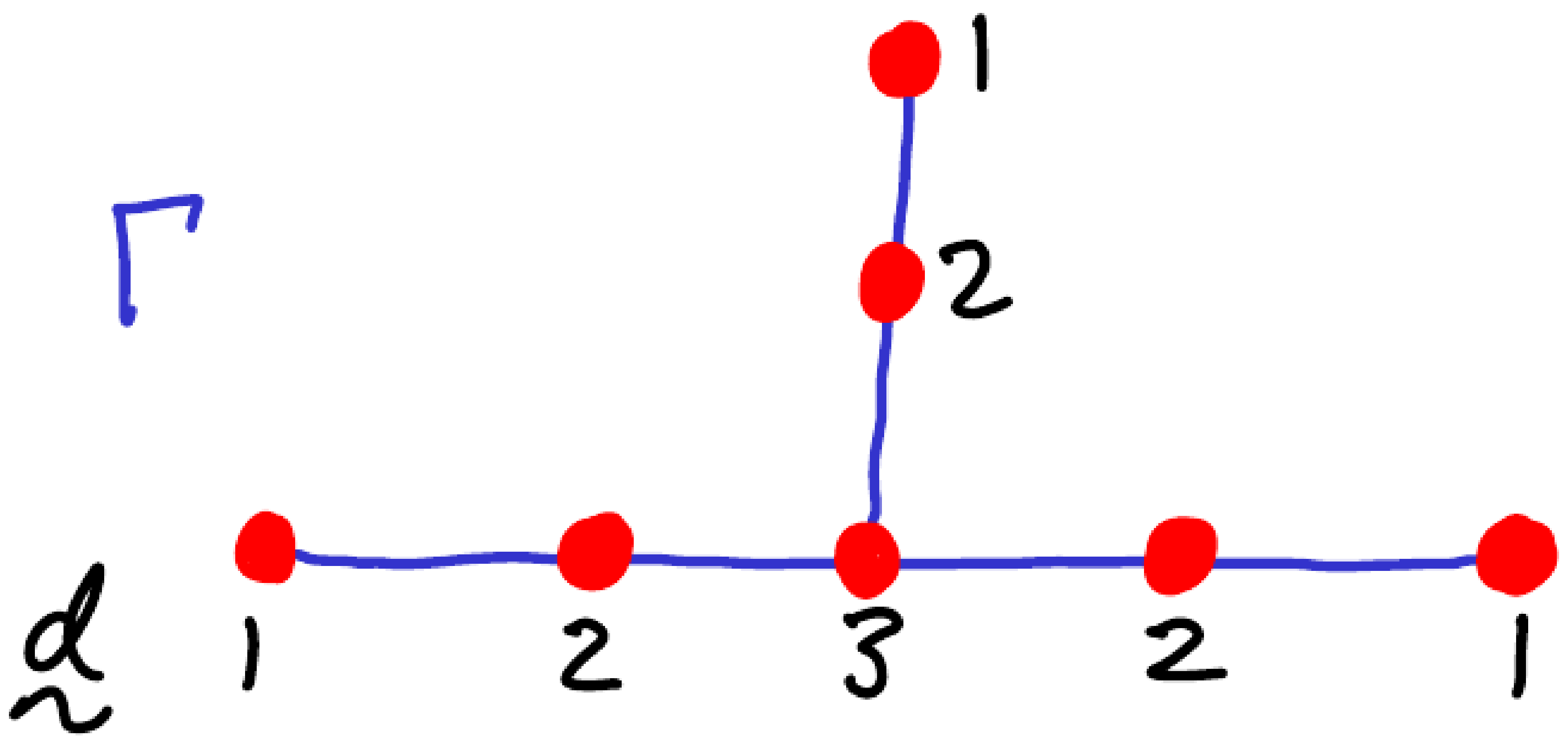

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Γ

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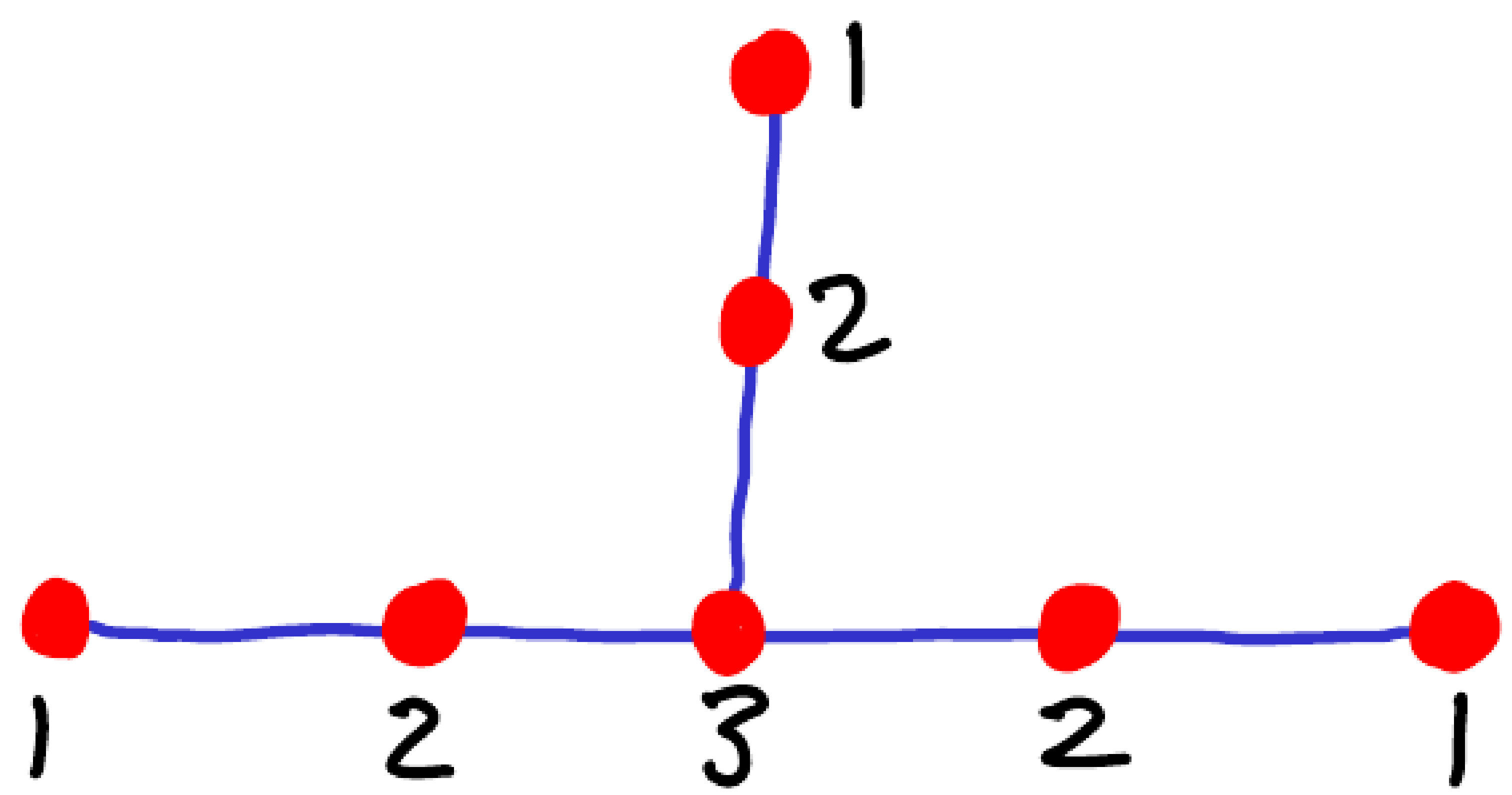

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$$\nabla = d - \left(\frac{A_1}{z-a_1} + \frac{A_2}{z-a_2} + \frac{A_3}{z-a_3} \right) dz \quad , \quad A_i \in \mathcal{O}_i$$

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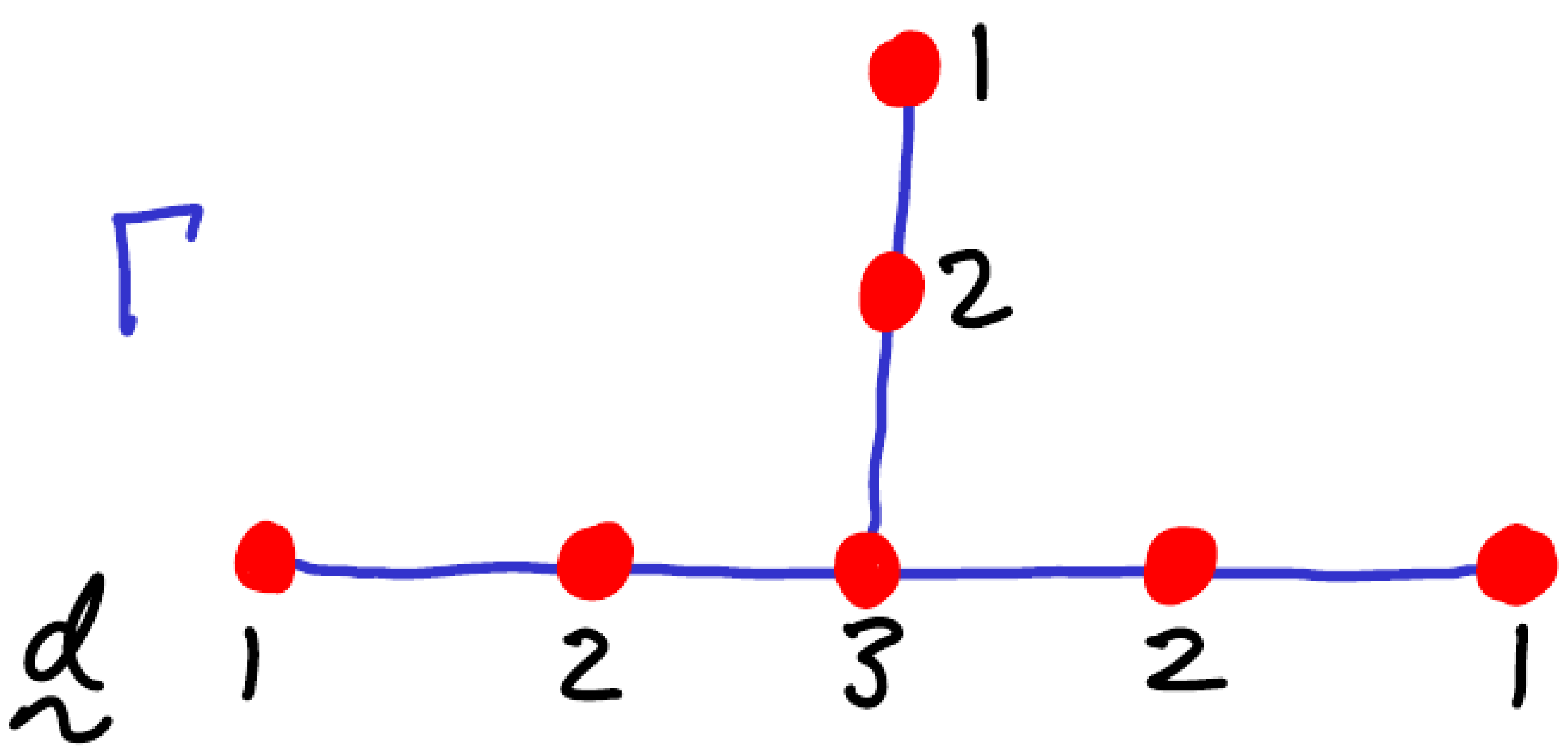

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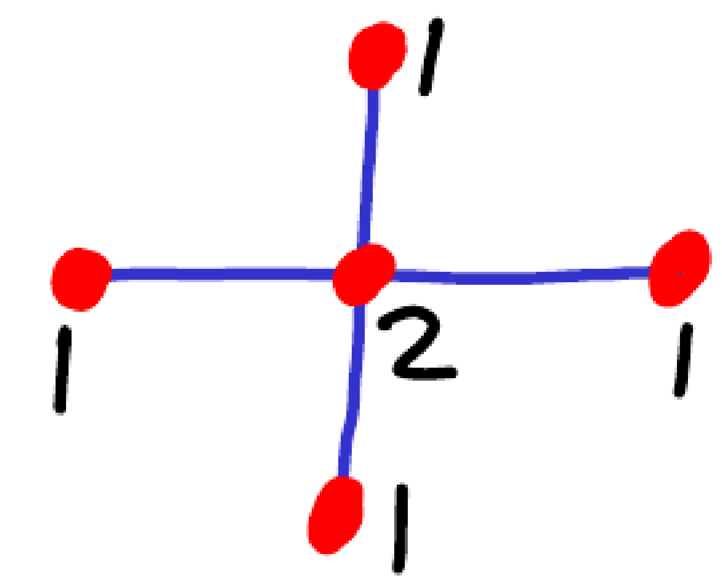
- NQV of any star-shaped Γ is modular (Kraft-Procesi, Nakajima, Crawley-Boevey)
- Get multiplicative version = character variety $\mathcal{M}_B \cong \mathcal{C}_1 \circledast \mathcal{C}_2 \circledast \mathcal{C}_3 /\!\!/ GL_3$

$\mathcal{M}^* \subset \mathcal{M}_{DR} \xrightarrow{RH} \mathcal{M}_B$ "Global Lie theory"

$\exists$ one more star-shaped affine Dynkin graph:

$\hat{D}_4$



$\sim$ quaternion group $\subset SU_2$

$\{\pm 1, \pm i, \pm j, \pm k\}$

$W(D_4) \circlearrowleft \mathbb{C}^4$ "constants"

Rank 2 Fuchsian systems with 4 poles $\rightsquigarrow$ cross ratio $\in \mathcal{M}_{0,4}$

"modular parameters" / "times"

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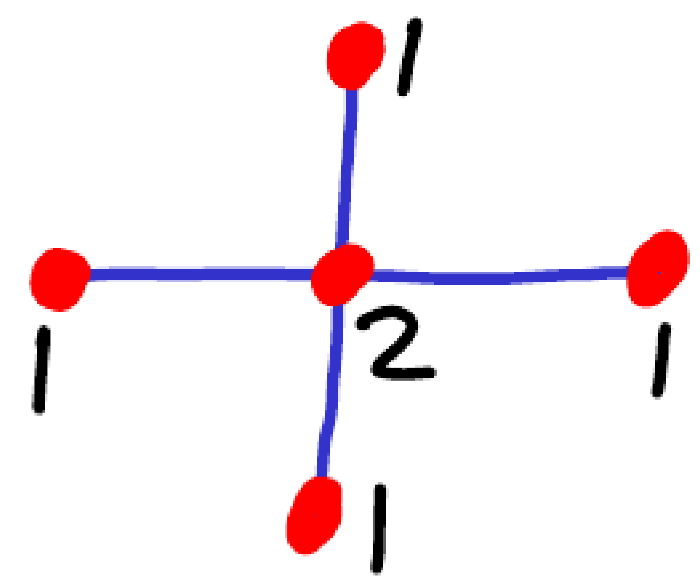

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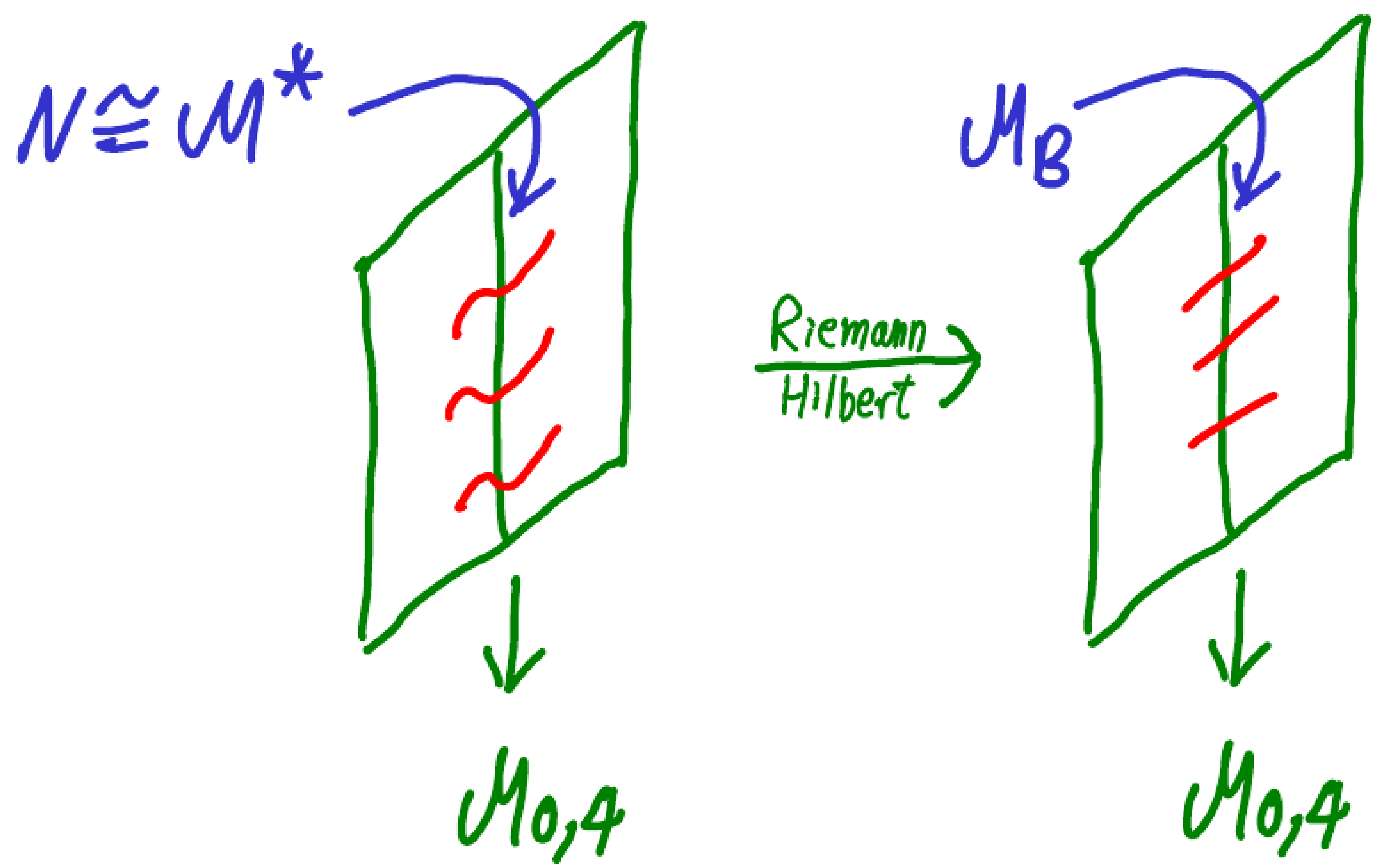
- Familiar from the Painlevé VI equation: (Richard Fuchs 1905)



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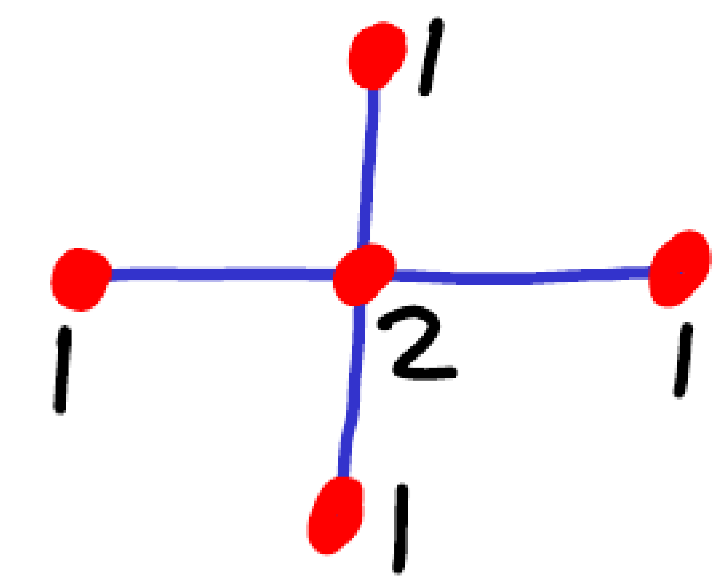

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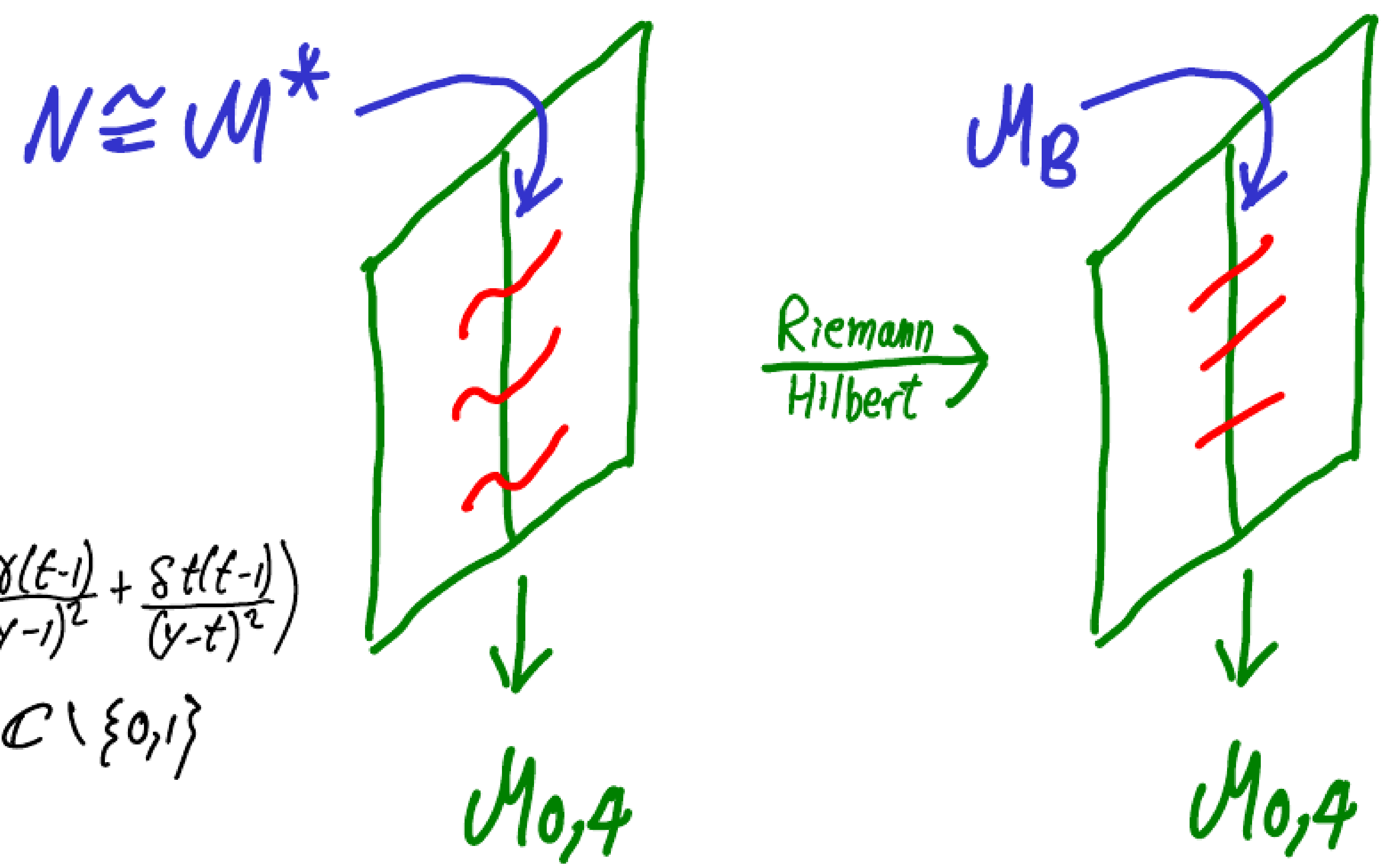
– Familiar from the Painlevé VI equation: (Richard Fuchs 1905)

$$y'' = \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\frac{(y')^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right)$$

$\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $t \in \mathcal{M}_{0,4} \cong \mathbb{C} \setminus \{0,1\}$

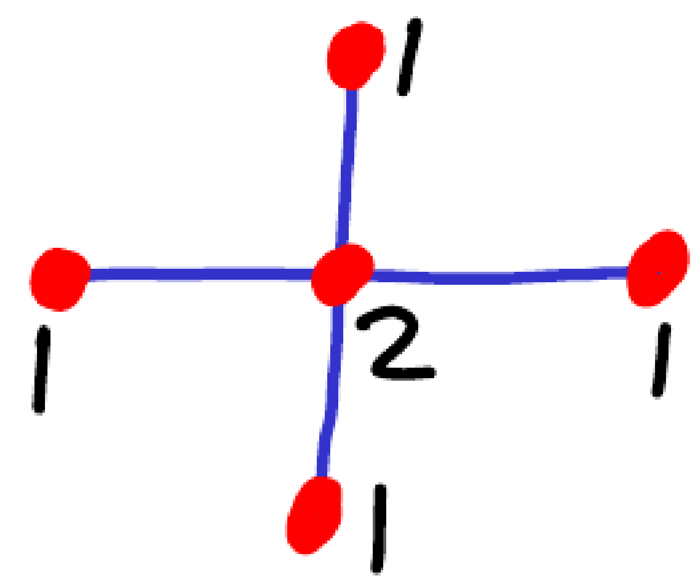


$\mathcal{M}_B \cong$ Fricke–Klein–Vogt cubic surface

$$xyz + x^2 + y^2 + z^2 = ax + by + cz + d$$

∃ one more star-shaped affine Dynkin graph:

$\hat{D}_4$



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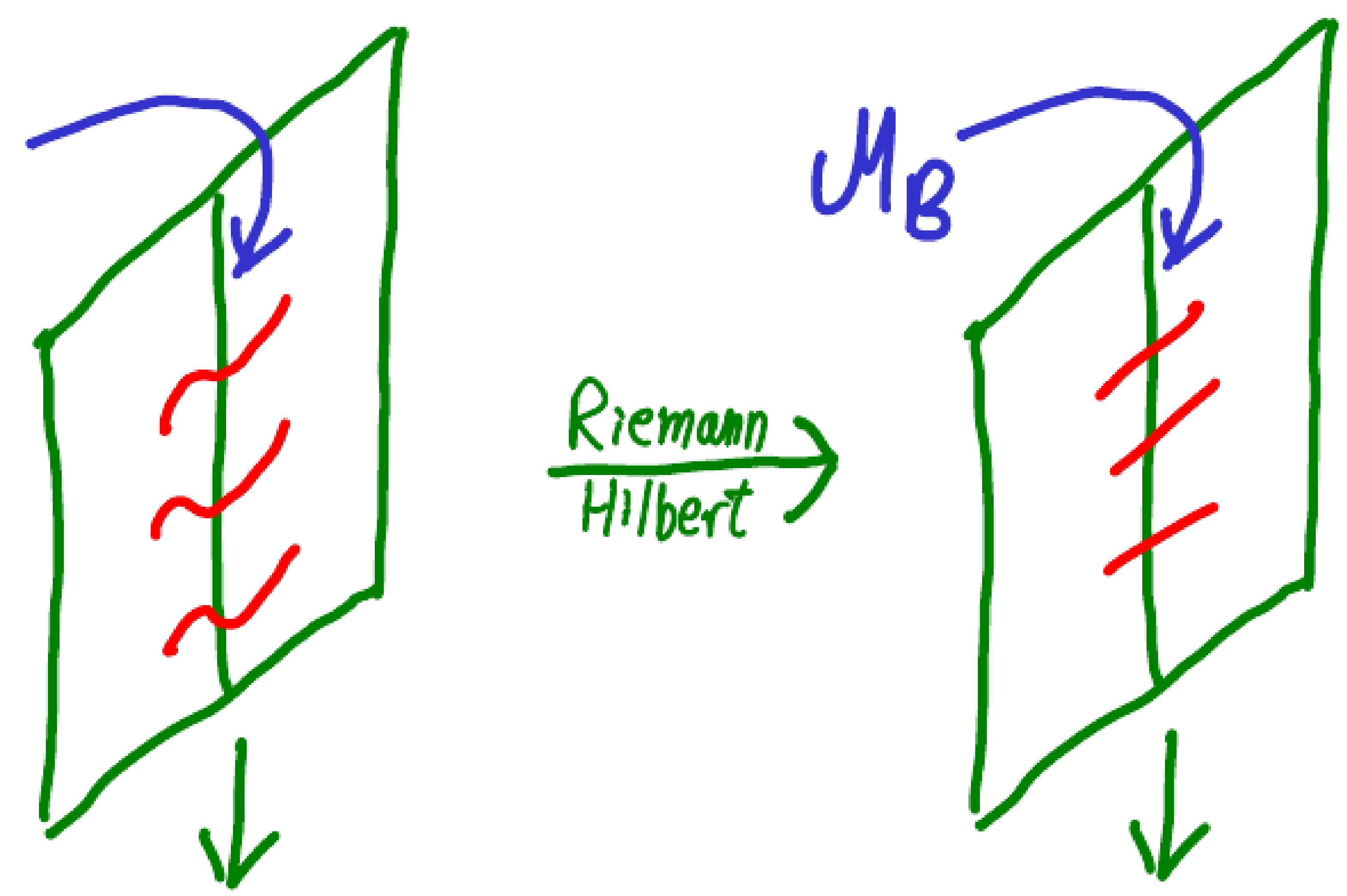
Rank 2 Fuchsian systems with 4 poles $\rightsquigarrow$ cross ratio $\in \mathcal{M}_{0,4}$

"modular parameters" / "times"

Okamoto 1987: affine Weyl group $W(\hat{D}_4) \circlearrowright \mathbb{C}^4$ relating $P_{\underline{VI}}$ equations

$N \cong \mathcal{M}^*$ $\qquad$ $\mathcal{M}_B$



$\mathcal{M}_{0,4}$ $\qquad$ $\mathcal{M}_{0,4}$

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THE PAINLEVÉ EQUATIONS AND THE DYNKIN DIAGRAMS

Kazuo Okamoto

Department of Mathematics
College of Arts and Sciences
University of Tokyo
Tokyo, Japan

1 Painlevé Systems

Let δ be a differential on $\mathbf{C}(t)$, i.e.

$$\delta = f(t)\frac{d}{dt},$$

$f(t)$ being a rational function in t, and

$$\mathrm{H}(t;q,p) \in \mathbf{C}[t,q,p],$$

a polynomial in three variables (t,q,p). We consider the Hamiltonian system of ordinary differential equations:

$$\begin{aligned} \delta q &= \frac{\partial \mathrm{H}}{\partial p}, \\ \delta p &= -\frac{\partial \mathrm{H}}{\partial q}, \end{aligned} \tag{1}$$

under the assumption that H is of the second degree with respect to p. Therefore, by

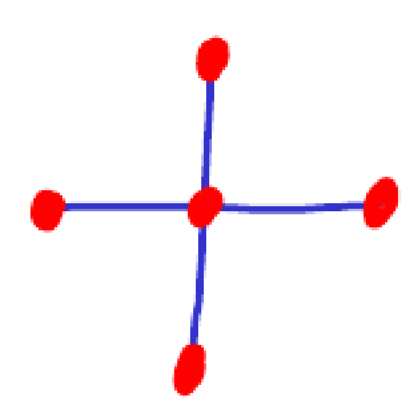
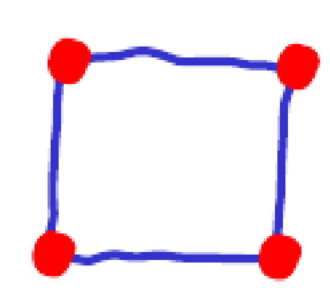
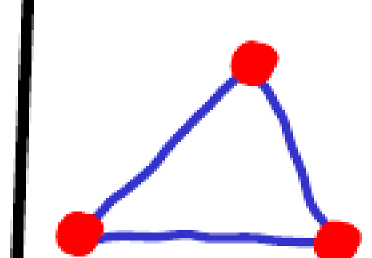
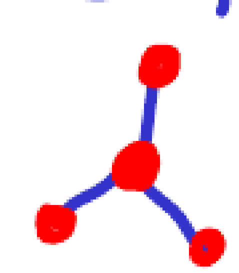
P_J	1	2	3	4	5	6
δ	$\frac{d}{dt}$	$\frac{d}{dt}$	$t\frac{d}{dt}$	$\frac{d}{dt}$	$t\frac{d}{dt}$	$t(t-1)\frac{d}{dt}$
number of parameters	0	1	2	2	3	4
Affine Weyl Group	--	$\mathbf{A}_1$	$\mathbf{B}_2$	$\mathbf{A}_2$	$\mathbf{A}_3$	$\mathbf{D}_4$
Particular solutions	--	Airy	Bessel	Hermite-Weber	Confluent Hyper-geometric	Gauß' Hyper-geometric

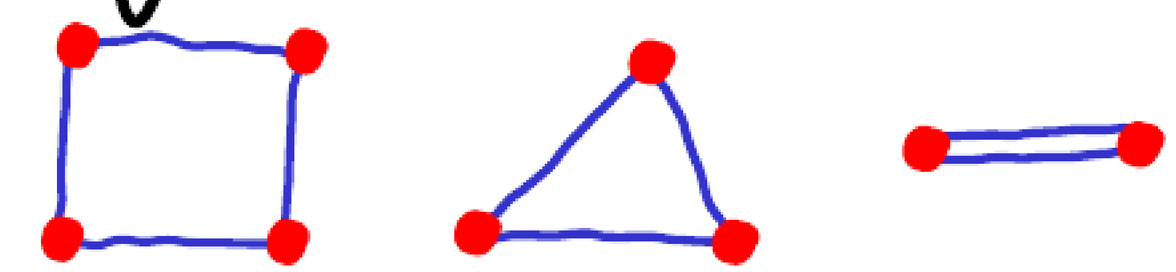
(0706.2634) Exercise 3: works for Painlevé 5, 4, 2 too:

$$\begin{cases} \mathcal{M}^* \cong NQV(\Gamma) & \text{(ALE space of type } \hat{A}_3, \hat{A}_2, \hat{A}_1\text{)} \\ \Gamma = \text{affine Dynkin graph of Okamoto symmetry group} \end{cases}$$

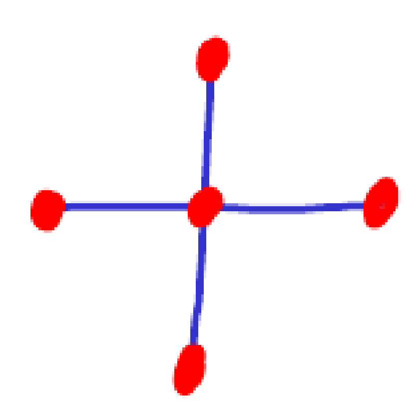
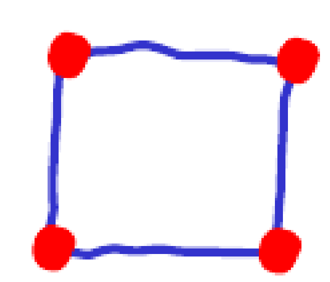
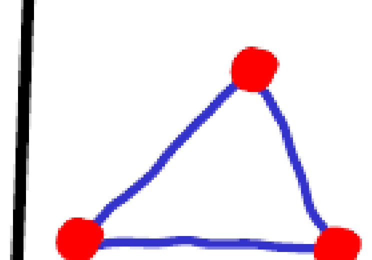
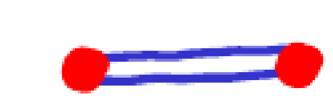
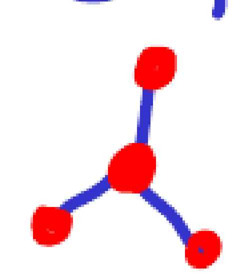
(0706.2634) Exercise 3: works for Painlevé 5, 4, 2 too:

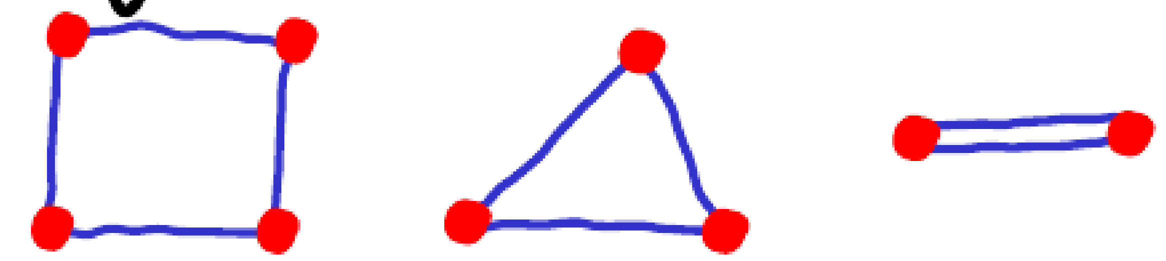
$$\begin{cases} \mathcal{M}^* \cong NQV(\Gamma) & \text{(ALE space of type } \hat{A}_3, \hat{A}_2, \hat{A}_1\text{)} \\ \Gamma = \text{affine Dynkin graph of Okamoto symmetry group} \end{cases}$$

Painlevé equation	6	5	4	3	2	1
pole orders ($g=0$, $rk\,2$)	1111	211	31	$22/11\tilde{2}$	$4/1\tilde{3}$	$\tilde{4}$
# constants	4	3	2	2	1	0
Diagram				?		?
Special solutions	Gauss $_2F_1$ 	Kummer $_1F_1$?	Weber ?	Bessel-Clifford $_0F_1$?	Airy ?	–

Questions ① What are the higher dimensional modular quiver varieties lying over  generalising the stars?

② What about Painlevé 1 & Painlevé 3 ($\mathcal{M}^*(P_3) \not\cong NQV(\Gamma)\ \forall\, \Gamma$) & their higher dimensional analogues?

Painlevé equation	6	5	4	3	2	1
pole orders $(g=0,\ rk\,2)$	1111	211	31	$22/11\tilde{2}$	$4/1\tilde{3}$	$\tilde{4}$
# constants	4	3	2	2	1	0
Diagram				?		?
Special solutions	Gauss $_2F_1$ 	Kummer $_1F_1$?	Weber ?	Bessel-Clifford $_0F_1$?	Airy ?	—

Questions (1) What are the higher dimensional modular quiver varieties lying over  generalising the stars?

(2) What about Painlevé 1 & Painlevé 3 ($\mathcal{M}^*(P_3) \not\cong NQV(\Gamma)\ \forall\, \Gamma$) & their higher dimensional analogues?

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(3) What is the 'deeper' analogue of $\mathcal{M}_{0,4}$ in general?

$\rightsquigarrow$ moduli of wild Riemann surfaces

(4) What is the 'deeper' analogue of the nonlinear local system $\mathcal{M}_B \to \mathcal{M}_{0,4}$?

$\rightsquigarrow$ local system of wild character varieties over any admissible deformation of a wild Riemann surface

[P.B. Annals of Math. 2014]

Choices: • $(\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$ "wild Riemann surface" (modular parameters)

• eigenvalues/orbit of Λ, parabolic weights (constants)

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Quiver modularity theorem

{ PB simply laced case + general conjecture
{ Hiroe–Yamakawa general proof

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"supernova graphs" (core + legs)

$$Q = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix}, \quad q_i \in x\,\mathbb{C}[x]$$

core nodes $= \{q_i\}$, $\#\text{edges}(q_i, q_j) = \deg(q_i - q_j) - 1$

+ legs from $\Lambda \in \mathfrak{h} = \prod \mathfrak{gl}_{d_i}(\mathbb{C})$